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Free Energy and Some Sample Path Properties of a Random Walk with Random Potential

Sergio Albeverio[1] **and Xian Yin Zhou**[2]

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We study the asymptotic behavior of the free energy for a model (defined by Sinai) of one-dimensional random walk with random potential. In particular, we obtain a central limit theorem and a strong law of large numbers for this free energy. We use some results on the free energy to study some sample path properties of this random walk which are related respectively to its recurrence and localization. Some exponents describing the recurrence and localization are found.

KEY WORDS: Random walks with random potentials; free energy; central limit theorem; recurrence; localization.

1. INTRODUCTION

To study the (physical) behavior of disordered systems several models have been developed. Models describing heteropolymers are of particular interest, also due to their importance in biology. In a model of this type first proposed by Garel *et al.*[4] the heteropolymer chain consists of two types of monomers: "hydrophobic" (A) and "hydrophile" (B), interacting with an (idealized) selective interface between water and another nonpolar solvent (e.g., oil). Garel *et al.* compare the model with experimental situations and discuss in physical terms a localization transition. This work is related to subsequent work by other authors. We cite here in particular the work by Grosberg *et al.*,[5] who provided a more detailed physical study of the localization transition for a simplified version of the model (see ref. 5 for

[1] Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum 1, Germany; BiBoS; SFB 237 Bochum-Essen-Düsseldorf; CERFIM, Locarno.

[2] Department of Mathematics, Beijing Normal University, Beijing 100875, China, and Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany.

further references). More precisely, ref. 5 investigated the behavior of the free energy near the point of transition from a delocalized to a localized regime for two basic models, chains with periodic resp. "annealed random" sequence links. Based on the work in refs. 4 and 5, Sinai[9] suggested a model of one-dimensional random walk with random potential, where the two links A and B appear as a Bernoulli environment, and established some precise sample path properties in this model. The main aim of the present paper is to study the asymptotic behavior of the free energy of Sinai's model and then discuss further sample properties. We will mainly prove that the central limit theorem holds for this free energy, and then use results on the free energy to discuss some path properties of this random walk, such as recurrence and localization, which are indeed different from the corresponding properties of the ordinary random walk. Concerning our results on the free energy, we had to develop our own methods, despite a large literature on the free energy of some other disordered systems (see, e.g., refs. 1, 3, 8, and 11 and references therein). In fact, it does not seem easy to adapt those methods to directly solve the problems posed by the present model. Now let us introduce the model we study in this paper.

Let $\{S_n\}_{n \geqslant 0}$ be a simple random walk in $\mathbf{Z}^1$, starting at the origin, on a probability space $(\Omega, \mathscr{F}, P)$, and the $b_k \in \{-1, 1\}$, $k \geqslant 0$, be independent identically distributed (i.i.d.) random variables on a probability space $(\tilde{\Omega}; \tilde{\mathscr{F}}, \tilde{P})$, which are independent of $\{S_n\}_{n \geqslant 0}$ on the product space $(\Omega \times \tilde{\Omega}, \mathscr{F} \otimes \tilde{\mathscr{F}}, P \otimes \tilde{P})$. For convenience, we let E and $\tilde{E}$ be, respectively, the expectations with respect to P and $\tilde{P}$, and assume $\tilde{E} b_k = 0$, $\forall k \geqslant 0$. As in ref. 9, let the function $U(x)$ be defined by

$$U(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

The partition function of this system is defined by

$$Z^{(0,n)} = E \exp\left(\beta \sum_{k=0}^{n} b_k U(S_k)\right), \qquad \forall n \geqslant 1$$

where $|\beta| \in (0, \infty)$ is a parameter representing the strength of the disorder. The free energy of this system is the random variable $\log Z^{(0,n)}$. Define a new probability measure by

$$P^{(0,n)}(A) = (Z^{(0,n)})^{-1} \int_A \exp\left(\beta \sum_{k=0}^{n} b_k U(S_k)\right) dP, \qquad \forall A \in \mathscr{F}$$

As mentioned in ref. 9, it is very interesting to study some properties such as recurrence and localization of this random walk. For some sequences $\{k_n\}_{n \geqslant 1}$ with $\lim_{n \to \infty} k_n = \infty$, Sinai[9] already obtained some estimates for the probability $P^{(0,n)}(S_{k_n} = x)$.

In this paper, both the central limit theorem and the strong law of large numbers are obtained for $\log Z^{(0,n)}$. Using these limit theorems, we prove some sample path properties (e.g., recurrence and localization) of this random walk $\{S_n\}_{n \geqslant 0}$ under the probability measure $P^{(0,n)}$. Unfortunately, we are still unable to show that the limit $\lim_{n \to \infty} P^{(0,n)}(S_k = x)$ exists with probability one for any fixed $k \geqslant 1$. Quite recently, Bolthausen and Hollander[12] discussed the localization–delocalization phase transition for the model with nonsymmetric random environment (i.e., $\tilde{E} b_k \neq 0$).

This paper is organized as follows. In Section 2 we study the asymptotic behavior of the mean value of this free energy. More precisely, we prove that there is a constant $\nu(\beta) \in (0, \infty)$ for $\beta \neq 0$ such that (see Proposition 2.6 below)

$$\left| \frac{\tilde{E} \log Z^{(0,n)}}{n} - \nu(\beta) \right| \leqslant O(1) \frac{\log n}{n}$$

The main aim of Section 3 is to prove that the variance of the free energy behaves roughly as n (for $n \to \infty$):

$$C_1 n \leqslant \operatorname{Var}(\log Z^{(0,n)}) \leqslant C_2 n, \qquad n \geqslant 1 \tag{1.1}$$

for some constants $C_1, C_2 \in (0, \infty)$. The upper bound in (1.1) can easily be proven by using the approach given in ref. 11. However, the proof of the lower bound in (1.1) is not so simple. In Section 3 we concentrate our main attention on the proof of this lower bound. In Section 4 we first prove that the variance of the free energy behaves exactly as $\gamma(\beta) n$ for some $\gamma(\beta) \in (0, \infty)$ with $\beta \neq 0$, and then prove that the central limit theorem holds for the free energy. Sections 5 and 6 are devoted to discussions of the sample path properties of the random walk $\{S_n\}$ under the probability measure $P^{(0,n)}$. We first discuss the asymptotic behavior of the quantity

$$\Delta_n =: \max_{1 \leqslant i < j \leqslant n} \{ j - i : S_i = S_j = 0, U(S_{i+1}) = \cdots = U(S_{j-1}) \neq 0 \}$$

The main result (see Theorem 5.3 below) is that there is a constant $\mu(\beta) \in (0, \infty)$ with $\beta \neq 0$ such that

$$P^{(0,n)}((\log n)^{-1} \Delta_n \in (\mu(\beta) + \varepsilon)^{-1}, (\mu(\beta) - \varepsilon)^{-1}) \xrightarrow{\tilde{P}} 1, \qquad n \to \infty$$

for all $\varepsilon \in (0, \mu(\beta))$. We remark that in the case of the simple random walk, Δ_n behaves roughly like n. In Section 6 we discuss the asymptotic behavior of $\max_{1 \leqslant i \leqslant n} |S_i|$ under the probability measure $P^{(0,n)}$. We prove that for any given $\beta \neq 0$ there is a constant $\chi(\beta) \in (0, \infty)$ such that the following holds for all $\varepsilon \in (0, 1)$ (see Theorem 6.1 below):

$$P^{(0,n)}((\log n)^{-1} \max_{1 \leqslant i \leqslant n} |S_i| \in [(1-\varepsilon)\chi(\beta), (1+\varepsilon)\chi(\beta)]) \xrightarrow{\tilde{P}} 1, \qquad n \to \infty$$

The following relation between $\mu(\beta)$ and $\chi(\beta)$ for $\beta \neq 0$ is also derived in Section 6:

$$\chi(\beta) = \begin{cases} (2\mu(\beta))^{-1/2}, & \mu(\beta) \in (0, 1/8] \\ (1/4 + 2\mu(\beta))^{-1}, & \mu(\beta) > 1/8 \end{cases}$$

For the case of the ordinary random walk in $\mathbf{Z}^1$, however, from the scaling property one can see that $\max_{1 \leqslant i \leqslant n} |S_i|$ behaves roughly as $n^{1/2}$. For the present random walk we obtain different relations between $\mu(\beta)$ and $\chi(\beta)$ for different $\mu(\beta)$. Since the present random walk has an exponential term as a statistical weight, its scaling property has been changed (we suggest looking at the proof of Lemma 6.2 for more information on this).

2. THE MEAN VALUE OF THE FREE ENERGY

Let $R_n = \tilde{E} \log Z^{(0,n)}$. The main aim of this section is to discuss the asymptotic behavior of R_n. It is easy to see that R_n satisfies asymptotically the subadditive property. Using this, we can easily show that the limit $\lim_{n \to \infty}(R_n/n)$ exists. We will derive a precise estimate for the partition function $Z^{(0,n)}$ (see Corollary 4.5 below). So we need to derive a more precise estimate for R_n/n (see Proposition 2.6 below). Let us first prove a lemma.

Lemma 2.1. For any given $\beta \neq 0$ there are constants ε, $\delta \in (0, \infty)$ such that

$$\tilde{P}\left(EI_{\{S_n = 0\}} \exp\left(\beta \sum_{i=0}^{n} b_i U(S_i)\right) \leqslant \exp(\delta n)\right) \leqslant O(1) \exp(-\varepsilon n)$$

if $n \geqslant 1$ is even.

The main idea of the following proof is basically from the proof of ref. 9, Lemma 1. Lemma 2.1 will be used to prove that the limit $\lim_{n \to \infty}(R_n/n)$ (if it exists) is positive.

Proof. Suppose $n \geqslant 1$ is even. Let $M = M(\beta) \geqslant 1$ be a fixed (even) number, whose value will be specified below. To prove this lemma we will mainly consider the special path of this random walk: $S_0 = 0$, $S_M = 0$, $S_{2M} = 0, \ldots, S_{nM} = 0$. For convenience we set

$$Z_{0,0}^{((i-1)M, iM)} = E \exp\left(\beta \sum_{j=1}^{M} b_{(i-1)M+j} U(S_j)\right) I_{\{S_{iM}=0\}}$$

and

$$\tilde{Z}^{(0,n)} = E I_{\{S_n=0\}} \exp\left(\beta \sum_{i=0}^{n} b_i U(S_i)\right)$$

Since $U(S_0) = 0$, we have

$$\begin{aligned}\tilde{Z}^{(0,nM)} &= \tilde{Z}^{(1,nM)} \\ &= E I_{\{S_{nM}=0\}} \exp\left(\beta \sum_{i=1}^{nM} b_i U(S_i)\right) \\ &\geqslant E \exp\left(\beta \sum_{i=1}^{nM} b_i U(S_i)\right) \prod_{i=1}^{n} I_{\{S_{iM}=0\}} \\ &\geqslant \prod_{i=1}^{n} Z_{0,0}^{((i-1)M, iM)} \end{aligned} \tag{2.1}$$

It is clear that $Z_{0,0}^{(0,M)}, Z_{0,0}^{(M,2M)}, \ldots, Z_{0,0}^{((n-1)M, nM)}$ are independent identically distributed random variables on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$. We want to prove that $\tilde{E} \log Z^{(0,n)} \geqslant 2$ if $M \geqslant 1$ is large enough. For this purpose we first prove that there are $M_0 \geqslant 1$ and $\delta_1(\beta) \in (0, \infty)$ such that

$$\tilde{E} \log Z_{0,0}^{(0,M)} \geqslant \delta_1(\beta)\, M^{1/2}, \qquad M \geqslant M_0 \tag{2.2}$$

Here and in the proof of Lemma 2.1 we always assume that M is even. By computation one can show that (see the proof of Lemma 3.4 below)

$$\begin{aligned} &P(S_1 > 0, \ldots, S_{M-1} > 0, S_M = 0) \\ &\quad = P(S_1 < 0, \ldots, S_{M-1} < 0, S_M = 0) \geqslant O(1)\, M^{-7/2} \end{aligned}$$

Thus, by the definition of $U(S_i)$ we have

$$\begin{aligned} Z_{0,0}^{(0,M)} &\geqslant E\Bigg(\exp\left(\beta \sum_{j=1}^{M} b_j U(S_j)\right) \\ &\qquad \times \left(I_{\{S_1>0,\ldots,S_{M-1}>0,\, S_M=0\}} + I_{\{S_1<0,\ldots,S_{M-1}<0,\, S_M=0\}}\right)\Bigg) \\ &\geqslant O(1)\, M^{-7/2} \left(\exp\left(\beta \sum_{i=1}^{M} b_i\right) + \exp\left(-\beta \sum_{i=1}^{M} b_i\right)\right) \end{aligned} \tag{2.3}$$

which implies that

$$\log Z_{0,0}^{(0,M)} \geqslant O(1) - \tfrac{7}{2}\log M + |\beta| \left| \sum_{i=1}^{M} b_i \right|$$

Hence, there are constants $C_1 \in \mathbf{R}^1$ and $C_2 \in (0, \infty)$ such that

$$\begin{aligned} \tilde{E} \log Z_{0,0}^{(0,M)} &\geqslant C_1 - \tfrac{7}{2}\log M + |\beta|\, \tilde{E} \left| \sum_{i=1}^{M} b_i \right| \\ &\geqslant C_1 - \tfrac{7}{2}\log M + C_2\, |\beta|\, M^{1/2}, \qquad M \geqslant 1 \end{aligned}$$

From this one can see that (2.2) is indeed true. Thus we can choose $M_0 \geqslant 1$ such that

$$\tilde{E} \log Z_{0,0}^{(0,M)} \geqslant 2, \qquad M \geqslant M_0$$

By (2.1) we know that

$$\log \tilde{Z}^{(0,nM)} \geqslant \sum_{i=1}^{n} \log Z_{0,0}^{((i-1)M,iM)}, \qquad M \geqslant M_0$$

and so

$$\tilde{E} \log \tilde{Z}^{(0,nM)} \geqslant 2n, \qquad M \geqslant M_0$$

By a large-deviation result we show that there is a constant $C(M) \in (0, \infty)$ such that

$$\begin{aligned} &\tilde{P}\left(\left| \sum_{i=1}^{n} (\log Z_{0,0}^{((i-1)M,iM)} - \tilde{E} \log Z_{0,0}^{((i-1)M,iM)} \right| \geqslant n \right) \\ &\quad \leqslant O(1) \exp(-C(M)n), \qquad \forall n \geqslant 1 \end{aligned}$$

This proves that

$$\tilde{P}\left(\log \tilde{Z}^{(0,nM)} \leqslant \sum_{i=1}^{n} \tilde{E} \log Z_{0,0}^{((i-1)M,iM)} - n \right) \leqslant O(1) \exp(-C(M)n)$$

which implies

$$\tilde{P}(\tilde{Z}^{(0,nM)} \leqslant e^n) \leqslant O(1) \exp(-C(M)n), \qquad \forall n \geqslant 1$$

Using this one can easily prove the desired result. ∎

Let P_x be the probability measure for the random walk $\{S_n\}_{n \geqslant 0}$ started at x, and E_x be the corresponding expectation. We remark that $P = P_0$ and $E = E_0$. The main result in this section is Proposition 2.6 below. To get this result we will mainly prove that R_n satisfies asymptotically a subadditive property (see, e.g., Lemma 2.3 below). To this end it suffices to prove that $E_0 \exp(\beta \sum_{i=1}^n b_i U(S_i))$ and $E_x \exp(\beta \sum_{i=1}^n b_i U(S_i))$ have asymptotically the same behavior. Lemmas 2.2 and 2.3 will be devoted to proving the above assertion.

Lemma 2.2. For any given $\beta \neq 0$ and $M \geqslant 1$ there are constants $K \geqslant 1$ and $C_3 \in (0, \infty)$ such that

$$\tilde{P}\left(E_x \exp\left(\beta \sum_{i=1}^n b_i U(S_i)\right) \geqslant m^K Z^{(0,n)}\right)$$
$$\leqslant C_3 m^{-M}, \qquad \forall x \in Z^1, \quad \forall m \geqslant 1$$

Before embarking on the rigorous proof of this lemma, let us make a few remarks. First, we need only prove the above estimate for $x \in \mathbf{Z}^1 \setminus \{0\}$. We introduce a random variable

$$\tau = \inf\{m \geqslant 0: S_m = 0\} \wedge n$$

Heuristically, we expect the asymptotic inequalities

$$E_x \exp\left(\beta \sum_{i=1}^{\tau} b_i U(S_i)\right) \leqslant E_0 \exp\left(\beta \sum_{i=1}^{\tau} b_i U(S_i)\right)$$
$$E_0 \exp\left(\beta \sum_{i=1}^{m} b_i U(S_i)\right) \leqslant E_0 \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right), \qquad m \leqslant n$$
$$\text{(by Lemma 2.1)}$$

to hold true, It is, on the other hand, not difficult to see that the lemma follows from these inequalities.

We now give the rigorous proof of Lemma 2.2.

Proof. By definition we know that $|x| \leqslant \tau \leqslant n$. Thus, by the Markov property of $\{S_i\}$ we have

$$E_x \exp\left(\beta \sum_{i=1}^n b_i U(S_i)\right)$$
$$= \sum_{j=|x|}^{n} E_x I_{\{\tau = j\}} \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) E \exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right) \qquad (2.4)$$

Since $b_1, ..., b_j$ are i.i.d. and $\tilde{E} b_i = 0$, by a large-deviation result we can show that for any given $\delta \in (0, 1)$ there is a constant $C_4 \in (0, \infty)$ such that

$$\tilde{P}\left(\sum_{i=1}^{j} b_i U(x) \geqslant bj\right) \leqslant O(1)\exp(-C_4 j)$$

On the other hand, by Lemma 2.1 we know that there are constants C_5, $C_6 \in (0, \infty)$ such that

$$\tilde{P}\left(E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}} \leqslant \exp(C_5 j)\right) \\ \leqslant O(1)\exp(-C_6 j)$$

if $j \geqslant 1$ is even. Hence, if $j \geqslant 1$ is even and $\delta \in (0, 1)$ is chosen to be sufficiently small,

$$\begin{aligned} &\tilde{P}\left(\exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) \geqslant E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}}\right) \\ &\quad \leqslant \tilde{P}\left(\exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) \geqslant \exp(\delta j)\right) \\ &\qquad + \tilde{P}\left(E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}} \leqslant \exp(\delta j)\right) \\ &\quad \leqslant O(1)\exp(-C_4 j) + \tilde{P}\left(E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) \leqslant O(1)\exp(C_5 j)\right) \\ &\quad \leqslant O(1)\exp(-C_4 \wedge C_6 j) \end{aligned}$$

Using the inequalities

$$\tilde{P}\left(\sum \xi_i \geqslant \sum \eta_i\right) < \tilde{P}\left(\bigcup \{\xi_i \geqslant \eta_i\}\right) \leqslant \sum \tilde{P}(\xi_i \leqslant \eta_i)$$

we show that

$$\begin{aligned} &\tilde{P}\left(\sum_{j=l[\log m]}^{n} E_x I_{\{\tau=j\}} \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) E\exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right)\right. \\ &\quad \left.\geqslant \sum_{j=l[\log m]}^{n} E_x I_{\{\tau=j\}} E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) E\exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right)\right) \\ &\quad \leqslant \sum_{j=l[\log m]}^{n} \tilde{P}\left(E_x I_{\{\tau=j\}} \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right)\right. \\ &\qquad\qquad \left.\geqslant E_x I_{\{\tau-j\}} E\exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}}\right) \end{aligned}$$

$$\leqslant O(1) \sum_{j=l[\log m]}^{n} \exp(-C_4 \wedge C_6 j)$$

$$\leqslant O(1) \exp(-C_7 l \log m) \tag{2.5}$$

for some constant $C_7 \in (0, \infty)$. Hence, if $l \geqslant 1$ is large enough and $j \geqslant l[\log m]$,

$$\text{l.h.s. of (2.5)} \leqslant O(1)\, m^{-M}$$

We also remark that

$$\sum_{j=l[\log m]}^{n} E_x I_{\{\tau=j\}} \leqslant 1$$

Using this, one can easily show that

$$\tilde{P}\left(\sum_{j=l[\log m]}^{n} E_x I_{\{\tau=j\}} \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) \times E \exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right) \geqslant Z^{(0,n)}\right)$$

$$\leqslant O(1)\, m^{-M} \tag{2.6}$$

if $l \geqslant 1$ is large enough. We now fix a constant $l \geqslant 1$ such that (2.6) holds. If j ($\leqslant l[\log m]$) is even, then there is a constant $K \geqslant 1$ such that

$$E \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) \leqslant O(1)\, m^{(K-1)/2}$$

and

$$E \exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}} \geqslant O(1)\, m^{-(K-1)/2} (\log m)^{-1/2}$$

Moreover, by the Markov property of $\{S_i\}$ we have

$$Z^{(0,n)} \geqslant E \exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}} E \exp\left(\beta \sum_{i=1}^{n-j} b_{j+i} U(S_i)\right)$$

Thus, by computation we get

$$\sum_{j=|x|}^{l[\log m]} E_x I_{\{\tau=j\}} \exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) E \exp\left(\beta \sum_{i=1}^{b-j} b_{i+j} U(S_i)\right)$$

$$\leqslant O(1) \sum_{j=|x|}^{l[\log m]} m^{(K-1)/2} E \exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right)$$

$$\leqslant O(1) \sum_{j=|x|}^{l[\log m]} m^{K-1} (\log m)^{1/2} E \exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j=0\}}$$

$$\times E \exp\left(\beta \sum_{i=1}^{n-j} b_{i+j} U(S_i)\right)$$

$$\leqslant O(1) \sum_{j=|x|}^{l[\log m]} m^{K-1} (\log m)^{1/2} E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right)$$

$$\leqslant O(1)\, m^K E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right)$$

Combining this estimate with (2.4) and (2.6), we then obtain the desired result. ▌

Lemma 2.3. For any given $\alpha \in (0, 1)$ there is a constant $C_8 \in (0, \infty)$ such that

$$R_{n+m} \leqslant C_8 \log n + R_n + R_m$$

if $n^\alpha \leqslant m \leqslant n$.

From Lemma 2.3 we can see that R_n is not strictly subadditive. However, the estimate given in Lemma 2.3 is enough for proving the existence of $\lim_{n \to \infty}(R_n/n)$. Lemma 2.3 is actually an immediate result of Lemma 2.2.

Proof. Since $|S_n| \leqslant n$, we have

$$R_{n+m} = \tilde{E} \log \sum_{x=-n}^{n} E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) I_{\{S_n=x\}}$$

$$\times E_x \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right)$$

By Lemma 2.2 we know that there is a constant $K_0 \geqslant 1$ such that

$$\tilde{P}\left(\max_{-n \leqslant x \leqslant n} E_x \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right) \geqslant n^{K_0} E \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right)\right)$$
$$\leqslant \sum_{x=-n}^{n} \tilde{P}\left(E_x \exp\left(\beta \sum_{i=1}^{m} b_i U(S_i)\right) \geqslant n^{K_0} E \exp\left(\beta \sum_{i=1}^{m} b_i U(S_i)\right)\right)$$
$$\leqslant O(1)\, n^{-4}$$

Thus, R_{n+m} is less than

$$\tilde{E} \log\left(n^{K_0} E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) E \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right)\right)$$
$$+ (\tilde{E}(\log Z^{(0,n)})^2)^{1/2} \tilde{P}^{1/2} \left(\max_{|x| \leqslant n} E_x \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right)\right.$$
$$\left.\geqslant n^{K_0} E \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right)\right)$$

which is bounded from above by

$$K_0 \log n + R_n + R_m + O(1)\, \beta n \cdot n^{-2} \leqslant C_9 \log n + R_n + R_m$$

for some constant $C_9 \in (0, \infty)$. This completes the proof of Lemma 2.3. ∎

We now use Lemma 2.3 to prove the existence of the limit $\lim_{n \to \infty}(R_n/n)$.

Lemma 2.4. For any given $\beta \neq 0$ there is a constant $\nu(\beta) \in (0, \infty)$ such that

$$\lim_{n \to \infty} n^{-1} R_n = \nu(\beta)$$

As mentioned before, R_n does not satisfy the (strict) subadditive property. So we cannot get the desired result immediately from Lemma 2.3. In the following we shall prove that for any given $\varepsilon \in (0, 1)$ and sufficiently large $m \geqslant 1$

$$n^{-1}(C_{10} \log n + R_n) \leqslant \varepsilon + m^{-1}(C_{10} \log m + R_m), \qquad n \geqslant m$$

which implies the existence of the limit $\lim_{n \to \infty}(R_n/n)$.

Proof. Let $\mu = \liminf_{n \to \infty}(R_n/n)$. By Lemma 2.3 we can show that there is a constant $C_{10} \in (0, \infty)$ such that

$$C_{10} \log(n+m) + R_{n+m} \leqslant C_{10} \log n + R_n + C_{10} \log m + R_m$$

if $n_\alpha \leqslant m \leqslant n$. Let

$$\tilde{R}_n = C_{10} \log n + R_n$$

Thus, if $n^\alpha \leqslant m \leqslant n$ and $n \geqslant 1$ is large enough,

$$\tilde{R}_{m+n} \leqslant \tilde{R}_n + \tilde{R}_m \tag{2.7}$$

It is clear that $\mu = \liminf_{n \to \infty}(\tilde{R}_n/n)$. For any given $\varepsilon \in (0, 1)$ there is a sufficiently large $m \geqslant 1$ such that

$$|m^{-1}\tilde{R}_m - \mu| \leqslant \varepsilon$$

Moreover, for any given $n \geqslant m$ there are $l_0,..., l_k \in \{0, 1\}$ and $m_0 \leqslant m$ such that $l_k = 1$ and

$$n = m \sum_{i=0}^{k} l_i 2^i + m_0$$

Let us set $R(u) = R_u$ and $\tilde{R}(u) = \tilde{R}_u$. From the proof of Lemma 2.3 we can see that for any fixed $L \in [1, k]$

$$R_n \leqslant O(1) \log n + R\left(m \sum_{i=k-L+1}^{k} l_i 2^i\right) + R\left(m \sum_{i=1}^{k-L} l_i 2^i + m_0\right) \tag{2.8}$$

Since $|b_i|$, $|U(x)| \leqslant 1$, we have

$$R_n \leqslant |\beta|\, n \tag{2.9}$$

We choose a sufficiently large constant $L \geqslant 1$ such that (for $L \leqslant k$)

$$\sum_{i=1}^{k-L} l_i 2^i \leqslant \varepsilon \sum_{i=1}^{k} l_i 2^i$$

By (2.9) we have

$$n^{-1} R\left(m \sum_{i=1}^{k-L} l_i 2^i + m_0\right) \leqslant \varepsilon\, |\beta| \tag{2.10}$$

Clearly, we can choose $\alpha \in (0, 1)$ and $k \geqslant L$ such that $m^\alpha \leqslant m/2$ and

$$2^{k-L+1} m \geqslant (2^k m)^\alpha$$

By (2.7) we have

$$\tilde{R}\left(m \sum_{i=k-L+1}^{k} l_i 2^i\right) \leqslant \sum_{i=k-L+1}^{k} l_i \tilde{R}(2^i m)$$

and

$$\tilde{R}(2^i m) \leqslant 2^i \tilde{R}_m$$

Therefore,

$$\tilde{R}\left(m \sum_{i=k-L+1}^{k} l_i 2^i\right) \leqslant \sum_{i=k-L+1}^{k} l^i 2^i \tilde{R}_m$$

From this we obtain that

$$\frac{\tilde{R}(m \sum_{i=k-L+1}^{k} l_i 2^i)}{n} \leqslant \frac{\sum_{i=k-L+1}^{k} l_i 2^i \tilde{R}_m}{\sum_{i=0}^{k} l_i 2^i m} \leqslant \frac{\tilde{R}_m}{m}$$

By (2.8) and (2.10) we know that

$$\frac{R_n}{n} \leqslant O(1)\, \frac{\log n}{n} + \varepsilon\, |\beta| + \frac{\tilde{R}_m}{m}$$

Since $\varepsilon \in (0, 1)$ is arbitrary, we have actually obtained that

$$\limsup_{n \to \infty} n^{-1} \tilde{R}_n \leqslant \liminf_{m \to \infty} m^{-1} \tilde{R}_m$$

which proves that the limit $\lim_{n \to \infty}(\tilde{R}_n/n)$ exists. Hence, $\lim_{n \to \infty}(R_n/n)$ exists. By Lemma 2.1 we know for some constant $\delta \in (0, \infty)$ that

$$R_n \geqslant \delta n, \qquad \forall n \geqslant 1$$

Thus, we know by (2.9) that $\lim_{n \to \infty} n^{-1} R_n \in (0, \infty)$. ∎

To get a more precise estimate for R_n, we first prove a lemma.

Lemma 2.5. For any given $\beta \neq 0$ there is a constant $C_{11} \in (0, \infty)$ such that

$$\nu(\beta) - \frac{R_n}{n} \leqslant C_{11}\, \frac{\log n}{n}, \qquad \forall n \geqslant 2$$

Proof. By (2.7) we know that if $m \geqslant 1$ is large enough,

$$\frac{\tilde{R}_{2m}}{2m} \leqslant \frac{\tilde{R}_m}{m} \tag{2.11}$$

For any given $n \geqslant m$ there is a constant $k(n) \geqslant 1$ such that

$$\left| v(\beta) - \frac{\tilde{R}(2^{k(n)}n)}{2^{k(n)}n} \right| \leqslant \frac{1}{n}$$

Thus, by (2.11) we have

$$\begin{aligned} v(\beta) - \frac{\tilde{R}_n}{n} &\leqslant \frac{\tilde{R}(2^{k(n)}n)}{2^{k(n)}n} - \frac{\tilde{R}_n}{n} + \frac{1}{n} \\ &\leqslant \sum_{i=1}^{k(n)} \left(\frac{\tilde{R}(2^i n)}{2^i n} - \frac{\tilde{R}(2^{i-1}n)}{2^{i-1}n} \right) + \frac{1}{n} \leqslant \frac{1}{n} \end{aligned}$$

which implies, by the relation between R_n and $\tilde{R}_n$, that

$$v(\beta) - \frac{R_n}{n} \leqslant O(1) \frac{\log n}{n}$$

if $n \geqslant 1$ is large enough. This completes the proof. ∎

Having these preparations, we can easily prove the main result in this section.

Proposition 2.6. For any given $\beta \neq 0$ there is a constant $C_{12} \in (0, \infty)$ such that

$$|v(\beta) - n^{-1} R_n| \leqslant C_{12} n^{-1} \log n, \qquad \forall n \geqslant 2$$

Proposition 2.6 is much more precise than Lemma 2.4. By means of Proposition 2.6 one can give a precise estimate on the partition function $Z^{(0,n)}$. It seems possible that the bound $n^{-1} \log n$ given in Proposition 2.6 can be replaced by n^{-1}.

Proof. By Lemma 2.5 we need only to show

$$v(\beta) - \frac{R_n}{n} \geqslant -C_{12} \frac{\log n}{n}, \qquad \forall n \geqslant 2$$

Let

$$\bar{R}_n = \tilde{E} \log \tilde{Z}^{(0,n)}$$

By a similar argument as in the proof of Lemma 2.5, we can show that there is a constant $\bar{v}(\beta) \in (0, \infty)$ such that $\bar{v}(\beta) = \lim_{n \to \infty} (\bar{R}_n / n)$ and

$$\bar{v}(\beta) - \frac{\bar{R}_n}{n} \geqslant -C_{13} \frac{1}{n}$$

for some constant $C_{13} \in (0, \infty)$. Thus, it suffices to prove

$$\frac{|\bar{R}_n - R_n|}{n} \leqslant O(1) \frac{\log n}{n}, \qquad \forall n \geqslant 2 \tag{2.12}$$

By a similar argument as in the proof of Lemma 2.2, we can show that for any given $M \geqslant 1$ there is a constant $K \geqslant 1$ such that

$$\tilde{P}\left(E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) I_{\{S_n \neq 0\}} \geqslant n^K \tilde{Z}^{(0,n)}\right)$$
$$\leqslant O(1)\, n^{-M}, \qquad \forall x \in \mathbf{Z}^1 \tag{2.13}$$

Thus, there is a constant $K_0 \geqslant 1$ such that

$$\tilde{P}\left(E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) I_{\{S_n \neq 0\}} \geqslant n^{K_0+1} \tilde{Z}^{(0,n)}\right)$$
$$\leqslant O(1)\, n^{-2}, \qquad \forall n \geqslant 1$$

Using this, we can show that

$$R_n \leqslant \tilde{E} \log((n^{K_0+1} + 1)\, \tilde{Z}^{(0,n)} + O(1)\, n^{-1}$$
$$\leqslant \bar{R}_n + O(1) \log n$$

It is clear that $\bar{R}_n \leqslant R_n$. Thus, we obtain the desired estimate (2.12). The proof of Proposition 2.6 is then complete. ∎

Remark. It would be very interesting to give an expression for the exponent $\nu(\beta)$ in terms of β. In fact, one can easily show that $\lim_{|\beta| \to \infty} \nu(\beta)/|\beta| = 1$, and $\nu(\beta) \leqslant \beta^2/2$ if $|\beta| > 0$ is sufficiently small.

3. A REASONABLE BOUND FOR THE VARIANCE

By a similar argument as for R_n given in Section 2 we can also show that the limit $\lim_{n \to \infty} n^{-1} \operatorname{Var}(\log Z^{(0,n)})$ exists (see, e.g., the proof of Lemma 4.1 below). However, we need to show that this limit is finite and positive. To this end, in this section we first derive a reasonable bound for the variance of the free energy [i.e., $\operatorname{Var}(\log Z^{(0,n)})$]. The main result in this section is as follows.

Proposition 3.1. Let $\beta \neq 0$; then there is a constant $C_1 \in (0, 1]$ such that

$$C_1 n \leqslant \operatorname{Var}(\log Z^{(0,n)}) \leqslant C_1^{-1} n, \qquad \forall n \geqslant 2$$

Wehr and Aizenman[11] already derived a reasonable bound for the variance of an extensive quantity (ref. 11, Definition 2.1). They also proved that the free energies of some disordered systems are indeed extensive quantities in their sense. Unfortunately, we were not able to find a random variable B_Λ for the system discussed in the present paper such that the conditions (2.2) and (2.4) in ref. 11 are satisfied. In other words, we were not able to prove in this way that the free energy $\log Z^{(0,n)}$ is an extensive quantity in the sense of ref. 11 Definition 2.1. The main result in ref. 11 (i.e., Theorem 2.3) thus cannot be used directly to prove Proposition 3.1 above (or Lemma 4.1 below), but some ideas given in the proof of ref. 11, Theorem 2.3, can be borrowed to prove Proposition 3.1 above.

For convenience, we set $l_n = [n/M]$, where $M \geqslant 1$ is large enough. Without loss of generality we may assume $l_n = n/M$. We introduce some notations, which correspond to those of ref. 11:

$$\psi_i = \sum_{j=(i-1)M+1}^{M} b_j U(S_j), \qquad i = 1,..., l_n$$

$$\mathscr{F}_{i,n} = \sigma\{b_{(i-1)M+1},..., b_{iM}\}, \qquad i = 1,..., l_n$$

and

$$\mathscr{F}'_{i,n} = \sigma\left(\bigcup_{1 \leqslant j \leqslant l_n, j \neq i} \{b_{(j-1)M+1},..., b_{jM}\}\right)$$

Let

$$Y_{i,n} = \tilde{E}(\log Z^{(0,n)}/\mathscr{F}'_{i,n})$$

and

$$Z_{i,n}(b_{(i-1)M+1},..., b_{iM}) = \tilde{E}(\log Z^{(0,n)}/\mathscr{F}_{i,n})$$

Then

$$Z_{i,n}(t_1,..., t_M) = \tilde{E} \log E \exp\left(\beta \sum_{1 \leqslant j \leqslant l_n, j \neq i} \psi_j + \beta \sum_{j=1}^{M} t_j U(S_{j+(i-1)M}\right)$$

For convenience, the random variable $Z_{i,n}(b_{(i-1)M+1},..., b_{iM})$ is sometimes briefly denoted by $Z_{i,n}$. Let $\{a_n\}_{n \geqslant 1}$ and $\{a'_n\}_{n \geqslant 1}$ be independent with respect to a probability measure P' and have the same distribution as $\{b_n\}_{n \geqslant 1}$ with respect to $\tilde{P}$. Then we can show that

$$\operatorname{Var}(Z_{i,n}) = \frac{1}{2} E'(Z_{i,n}(a_{(i-1)M+1},..., a_{iM}) - Z_{i,n}(a'_{(i-1)M+1},..., a'_{iM}))^2$$

$$= \frac{1}{2} E' \left(\tilde{E} \log \frac{E \exp(\beta \sum_{1 \leqslant j \leqslant l_n, j \neq i} \psi_j + \beta \sum_{j=(i-1)M+1}^{iM} a_j U(S_j))}{E \exp(\beta \sum_{1 \leqslant j \leqslant l_n, j \neq i} \psi_j + \beta \sum_{j=(i-1)M+1}^{iM} a'_j U(S_j))} \right)^2 \tag{3.1}$$

where E' is the expectation with respect to P'. Let us first prove a lemma.

Lemma 3.2. We have the following relation:

$$\sum_{i=1}^{l_n} \operatorname{Var}(Z_{i,n}) \leqslant \operatorname{Var}(\log Z^{(0,n)}) \leqslant \sum_{i=1}^{l_n} \operatorname{Var}(Y_{i,n})$$

Proof. We only give a sketch of the proof (for more details, the reader is referred to the proof of ref. 11, Proposition 3.1). Define the map Q_i by

$$Q_i \log Z^{(0,n)} = \tilde{E}(\log Z^{(0,n)} / \mathscr{F}'_{i,n})$$

Let

$$\mathscr{P}_i = \prod_{j=1}^{i} Q_i, \qquad \mathscr{P}_0 = I$$

Then one can show that

$$Z_{i,n} = \prod_{1 \leqslant j \leqslant l_n, j \neq i} Q_j \log Z^{(0,n)}$$

and

$$\tilde{E}(\mathscr{P}_i \log Z^{(0,n)} - \mathscr{P}_{i-1} \log Z^{(0,n)})(\mathscr{P}_j \log Z^{(0,n(} - \mathscr{P}_{j-1} \log Z^{(0,n)}) = 0, \qquad i \neq j$$

Using these results, we obtain that

$$\operatorname{Var}(Z_{i,n}) = \tilde{E} \left(\prod_{1 \leqslant j \leqslant l_n, j \neq i} Q_j (I - Q_i) \log Z^{(0,n)} \right)^2$$

and

$$\operatorname{Var}(\log Z^{(0,n)}) = \sum_{i=1}^{l_n} \tilde{E}((\mathscr{P}_{i-1} - \mathscr{P}_i) \log Z^{(0,n)})^2$$

$$= \sum_{i=1}^{l_n} \tilde{E}(\mathscr{P}_{i-1}(I - Q_i) \log Z^{(0,n)})^2$$

Using the Hölder inequality for conditional expectations, we have

$$\tilde{E}\left|\prod_{1\leqslant j\leqslant l_n, j\neq i} Q_j(I-Q_j)\log Z^{(0,n)}\right|^2$$
$$\leqslant \tilde{E}\left|\prod_{1\leqslant j\leqslant l_n-1, j\neq i} Q_j(I-Q_i)\log Z^{(0,n)}\right|^2$$
$$\leqslant \tilde{E}\,|\mathscr{P}_{i-1}(I-Q_i)\log Z^{(0,n)}|^2$$

Therefore

$$\sum_{i=1}^{l_n} \operatorname{Var}(Z_{i,n})\leqslant \operatorname{Var}(\log Z^{(0,n)})$$

which proves the lower bound. By a similar argument we can prove the upper bound. ∎

In Lemma 3.2 we may choose $l_n=[M^{-1}n]$. It is easy to show that $\{\operatorname{Var}(Y_{i,n})\}$ is bounded (see the proof of Proposition 3.1 below). Thus we can easily get the upper bound asserted in Proposition 3.1 from Lemma 3.2. To get the lower bound from Lemma 3.2, we have to prove that $\operatorname{Var}(Z_{i,n})$ has a uniform lower bound. For this purpose we let $\{t_n\}_{n\geqslant 1}$ and $\{t'_n\}_{n\geqslant 1}$ be two special sequences:

$$t_k=1,\quad t'_{2k-1}=-1,\quad t'_{2k}=1,\qquad \forall k\geqslant 1$$

Let us first prove two lemmas.

Lemma 3.3. For any given $\beta\neq 0$ the following holds for any $x\in Z^1$:

$$E_x\exp\left(\beta\sum_{j=1}^{M} t_jU(S_j)\right)\leqslant O(1)\exp(|\beta|\,M/2)$$

Proof. Let $\sigma_1=\inf\{j\geqslant 1:\ S_j=0\}$ and $\sigma_2=\sup\{j\leqslant M:\ S_j=0\}$. Since $|\sum_{j=1}^{u} t'_j|\leqslant 1$, $\forall u\geqslant 1$, by the definitions of σ_1 and σ_2 we have

$$E_x\exp\left(\beta\sum_{j=1}^{M} t'_jU(S_j)\right)$$
$$=E_xI_{\{\sigma_1\leqslant M,\sigma_2\geqslant 1\}}\exp\left(\beta\sum_{j=\sigma_1+1}^{\sigma_2} t'_jU(S_j)\right)$$
$$\times\exp\left(\beta\sum_{j=1}^{\sigma_1} t'_jU(x)+\beta\sum_{j=\sigma_2+1}^{v_n} t'_jU(y)\right)+\exp\left(\beta\sum_{j=1}^{M} t'_jU(x)\right)$$
$$\leqslant e^{\beta}+e^{2\beta}\max_{1\leqslant i\leqslant M} E\exp\left(\beta\sum_{j=1}^{i} t'_jU(S_j)\right)I_{\{S_i=0\}}$$

If $i \leqslant M/2$, then [since $t_j U(S_j) \leqslant 1$]

$$E \exp\left(\beta \sum_{j=1}^{i} t_j' U(S_j)\right) \leqslant \exp(|\beta|\ M/2)$$

We now assume $M \geqslant i \geqslant M/2$ and set $L_i = \sum_{j=1}^{i} I_{\{S_j = 0\}}$. If $L_i \geqslant i/2$, then we also have

$$E \exp\left(\beta \sum_{j=1}^{i} t_j' U(S_j)\right) \leqslant E \exp(|\beta|\ i/2) \leqslant \exp(|\beta|\ M/2)$$

Now suppose $L_i \leqslant i/2$ and set

$$\{s(1),..., s(L_i)\} = \{j \leqslant i\colon S_j = 0\}, \qquad s(0) = 0$$

It is clear that

$$U(S_j) = U(S_{s(v)+1}), \qquad j \in [s(v)+1, s(v+1)-1]$$

By the definition of $\{t_j'\}_{j \geqslant 1}$ we know that

$$\left| \sum_{j=s(v)+1}^{s(v+1)} t_j' U(S_j) \right| \leqslant 1$$

Thus

$$\begin{aligned} \left| \sum_{j=1}^{i} t_j' U(S_j) \right| &= \left| \sum_{v=1}^{L_i} \sum_{j=s(v-1)+1}^{s(v)} t_j' U(S_j) \right| \\ &\leqslant \sum_{v=1}^{L_i} \left| \sum_{j=s(v-1)+1}^{s(v)} t_j' U(S_j) \right| \\ &\leqslant L_i \leqslant M/2 \end{aligned}$$

In all cases we have

$$E \exp\left(\beta \sum_{j=1}^{i} t_j' U(S_j)\right) \leqslant \exp(|\beta|\ M/2)$$

which proves the desired result. ∎

Lemma 3.4. For any given $\beta \neq 0$ the following holds:

$$\begin{aligned} &E_x \exp\left(\beta \sum_{j=1}^{M} t_j U(S_j)\right) I_{\{S_M = y\}} \\ &\qquad \geqslant O(1)\, M^{-7/2} \exp(|\beta|\ (M - 2\,|x| - 2\,|y|))\, 2^{-|x| - |y|} \end{aligned} \tag{3.2}$$

if $|x| + |y| \leqslant M$ and the left-hand side of (3.2) is positive.

By (3.2) we known that if $|x|+|y|$ is less than εM with $\varepsilon \in (0, 1/16)$, then the left-hand side (l.h.s.) of (3.2) is bigger than $\exp(3\,|\beta|\, M/4)$. The idea to prove (3.2) is also simple. To get a lower bound for the l.h.s. of (3.2), we only need to consider some special path of the random walk $\{S_i\}$ (e.g., the path $\{S_1 = x - U(x),..., S_{|x|-1} = U(x),\ S_{|x|} = 0;\ S_{M-|y|} = 0,..., S_{M-1} = y - U(y),\ S_M = y\}$).

Proof. We assume that the l.h.s. of (3.2) is positive. Without loss of generality we may assume $\beta > 0$ and $x, y \neq 0$. It is easy to see from the Markov property of $\{S_i\}$ that

$$
\begin{aligned}
&E_x \exp\left(\beta \sum_{j=1}^{M} t_j U(S_j)\right) I_{\{S_M = y\}} \\
&\geqslant E_x \Bigg[\exp\left(\beta \sum_{j=|x|+1}^{M-|y|} t_j U(S_j) + \beta \sum_{j=1}^{|x|} t_j U(x) + \beta \sum_{j=M-|y|+1}^{M} t_j U(y)\right) \\
&\qquad \times I_{\{S_1 = x - U(x),..., S_{|x|-1} = U(x),\ S_{|x|} = 0;\ S_{M-|y|} = 0,..., S_{M-1} = y - U(y),\ S_M = y\}} \Bigg] \\
&\geqslant 2^{-|x|-|y|} e^{-\beta|x| - \beta|y|} E \exp\left(\beta \sum_{j=1}^{M-|y|-|x|} t_{j+|x|} U(S_j)\right) I_{\{S_{M-|x|-|y|} = 0\}}
\end{aligned}
\tag{3.3}
$$

Let $\tau_z = \inf\{i \geqslant 1\colon S_i = z\}$ and $\gamma_k = \inf\{m \geqslant 1\colon |S_m| \geqslant k\}$. Both τ_z and γ_k are stopping times. It is easy to show that

$$
P_1(\gamma_{[m^{3/4}]} < \tau_0) \geqslant O(1)\, m^{-3/4}
$$

By the symmetry property of $\{S_i\}$ we have

$$
\begin{aligned}
&P(S_1 > 0,..., S_m > 0) \\
&\quad = \sum_{x=1}^{m} P(S_1 > 0,..., S_{m-1} > 0, S_m = x) \\
&\quad \geqslant P(S_1 > 0,..., S_m > 0, \max_{1 \leqslant i \leqslant m} S_i \leqslant m^{3/4}) \\
&\quad \geqslant \tfrac{1}{2} P_1(\gamma_{[m^{3/4}]} < \tau_0) \\
&\quad \geqslant O(1)\, m^{-3/4}
\end{aligned}
$$

This proves that there is $x_0 \in [1, m]$ such that

$$
P(S_1 > 0,..., S_{m-1} > 0, S_m = x_0) \geqslant O(1)\, m^{-1-3/4}
$$

By the Markov property of $\{S_i\}$ we can show that

$$
\begin{aligned}
&P(S_1>0,\dots,S_{2m-1}>0,\ S_{2m}=0)\\
&\quad=\sum_{x=1}^{m} P(S_1>0,\dots,S_m=x,\ S_{m+1}>0,\dots,S_{2m}=0)\\
&\quad=\sum_{x=1}^{m} P(S_1>0,\dots,S_m=x)\,P(S_1>-x,\dots,S_{m-1}>-x,\ S_m=-x)\\
&\quad=\sum_{x=1}^{m} P^2(S_1>0,\dots,S_{m-1}>0,\ S_m=x)\\
&\quad\geqslant P^2(S_1>0,\dots,S_{m-1}>0,\ S_m=x_0)\\
&\quad\geqslant O(1)\,m^{-7/2}
\end{aligned}
$$

By (3.3) we know that

$$
\begin{aligned}
&E_x\exp\left(\beta\sum_{j=1}^{M} t_jU(S_j)\right)I_{\{S_{v_n}=y\}}\\
&\quad\geqslant 2^{-|x|-|y|}e^{-\beta|x|-\beta|y|}\exp(\beta(M-|y|-|x|-1)\\
&\qquad\times P(S_1>0,\dots,S_{M-|y|-|x|-1}>0,\ S_{M-|y|-|x|}=0)\\
&\quad\geqslant O(1)(M-|y|-|x|)^{-7/2}2^{-|x|-|y|}e^{\beta(M-2|x|-2|y|-1)}
\end{aligned}
$$

which proves the desired result. ▮

Let

$$
\phi_1(i)=E\exp\left(\beta\sum_{1\leqslant j\leqslant l_n,\,j\neq i}\psi_j+\beta\sum_{j=(i-1)M+1}^{iM} t_jU(S_j)\right)
$$

and

$$
\phi_2(i)=E\exp\left(\beta\sum_{1\leqslant j\leqslant l_n,\,j\neq i}\psi_j+\beta\sum_{j=(i-1)M+1}^{iM} t'_jU(S_j)\right)
$$

Lemma 3.5. For any given $\beta\neq 0$ there are constants $\delta_1,\delta_2\in(0,\infty)$ such that

$$
\tilde{P}(\phi_1(i)\leqslant\exp(\delta_1 M)\,\phi_2(i))\leqslant O(1)\exp(-\delta_2 M),\qquad \forall n\geqslant 1,\quad \forall M\geqslant 1
$$

Before proving Lemma 3.5, let us remark that from Lemmas 3.3 and 3.4 we can expect that the inequality

$$
E_x\exp\left(\beta\sum_{j=1}^{M} t'_jU(S_j)\right)\leqslant e^{-|\beta|M/4}E_x\exp\left(\beta\sum_{k=1}^{M} t_jU(S_j)\right)I_{\{S_M=y\}}
$$

holds, which is heuristically at least compatible with the conclusion of Lemma 3.5. However, the rigorous proof of Lemma 3.5 still involves some more complicated computations, as we shall now see.

Proof. Since $S_{(i-1)M}$, $S_{iM} \in \mathbf{Z}^1$, by the Markov property we know that

$$\phi_2(i) = \sum_{x, y \in \mathbf{Z}^1} E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = x\}}$$
$$\times E_x \exp\left(\beta \sum_{j=1}^{M} t'_{j+(i-1)M} U(S_j)\right)$$
$$\times E_y \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right)$$

We remark that there are constants $\delta_3, \delta_4 > 0$ such that (see the proof of Lemma 2.2)

$$\tilde{P}\left(\exp\left(\beta \sum_{i=1}^{j} b_i U(x)\right) \exp(\delta_3 j) \geqslant E \exp\left(\beta \sum_{i=1}^{j} b_i U(S_i)\right) I_{\{S_j = 0\}}\right)$$
$$\leqslant O(1) \exp(-\delta_4 j)$$

if $j \geqslant 1$ is even. Thus, as in the derivation of (2.5), we can show that there are constants δ_5, $\delta_6 \in (0, \infty)$ such that

$$\tilde{P}(A_i(x)) \leqslant O(1) \exp(-\delta_6 |x|)$$
$$\tilde{P}(B_i(y)) \leqslant O(1) \exp(-\delta_6 |y|)$$

if $(i-1)M$ is even, where

$$A_i(x) = \left\{E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = x\}}\right.$$
$$\left.\geqslant \exp(-\delta_5 |x|)\, E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = 0\}}\right\}$$
$$B_i(y) = \left\{E_y \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right)\right.$$
$$\left.\geqslant \exp(-\delta_5 |y|)\, E \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right)\right\}$$

Therefore, there is $\delta_7 \in (0, \infty)$ such that

$$\tilde{P}\left(\bigcup_{|x|,\,|y| \geqslant \varepsilon M} A_i(x) \cup B_i(y)\right) \leqslant O(1) \exp(-\delta_7 \varepsilon M), \qquad \forall \varepsilon \in (0, 1)$$

If the following event has occurred

$$\bigcap_{|x| \geqslant \varepsilon M,\, |y| \geqslant \varepsilon M} A_i^c(x) \cap B_i^c(y) \tag{3.4}$$

by Lemma 3.3 it follows that

$$\begin{aligned}
\phi_2(i) \leqslant O(1) & \sum_{|x|,\,|y| \leqslant \varepsilon M} E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = x\}} \\
& \times \exp(|\beta|\, M/2)\, E_y \left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right) \\
& + O(1) \sum_{x \in Z^1,\, |y| \geqslant \varepsilon M} E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = x\}} \\
& \times \exp(|\beta|\, M/2) \exp(-\delta_5\, |y|)\, E \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right) \\
& + O(1) \sum_{|x| \geqslant \varepsilon M,\, |y| \leqslant \varepsilon M} E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{S_{(i-1)M} = 0\}} \\
& \times \exp(-\delta_5\, |x|) \exp(|\beta|\, M/2) E_y \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right) \\
\leqslant {} & O(1) \exp(|\beta|\, M/2) E \exp\left(\beta \sum_{j=1}^{i-1} \psi_j\right) I_{\{|S_{(i-1)M}| \leqslant \varepsilon M\}} \\
& \times \sum_{|y| \leqslant \varepsilon M} E_y \exp\left(\beta \sum_{j=1}^{n-iM} b_{j+iM} U(S_j)\right)
\end{aligned}$$

By Lemma 3.4 we know that if $\varepsilon \in (0, |\beta|)$ is sufficiently small and $|x|$, $|y| \leqslant \varepsilon M$, then

$$E_x \exp\left(\beta \sum_{j=1}^{M} t_j U(S_j)\right) I_{\{S_M = y\}} \geqslant O(1) \exp(3\, |\beta|\, M/4)$$

Hence, if the event (3.4) has occurred, by the Markov property of $\{S_i\}$ we get

$$\begin{aligned}\phi_1(i) &\geqslant \sum_{|x|,\,|y|\leqslant \varepsilon M} E\exp\left(\beta\sum_{j=1}^{i-1}\psi_j\right) I_{\{S_{(i-1)M}=x\}}\\&\quad\times E^x\exp\left(\beta\sum_{j=1}^{M} t_{j+(i-1)M}U(S_j)\right) I_{\{S_M=y\}}\\&\quad\times E_y\exp\left(\beta\sum_{j=1}^{n-iM} b_{j+iM}U(S_j)\right)\\&\geqslant O(1)E\exp\left(\beta\sum_{j=1}^{i-1}\psi_j\right) I_{\{|S_{(i-1)M}|\leqslant \varepsilon M\}}\\&\quad\times\exp(3\,|\beta|\,M/4)\sum_{|y|\leqslant\varepsilon M} E_y\exp\left(\beta\sum_{j=1}^{n-iM} b_{j+iM}U(S_j)\right)\\&\geqslant O(1)\exp(|\beta|\,M/4)\,\phi_2(i)\end{aligned}$$

Using this, we prove that there is a constant $C_2\in(0,\infty)$ such that

$$\begin{aligned}\tilde{P}(\phi_1(i)&\leqslant C_2\exp(|\beta|\,M/4)\,\phi_2(i))\\&\leqslant\tilde{P}\left(\bigcup_{|x|,\,|y|\geqslant\varepsilon M} A_i(x)\cup B_i(y)\right)\\&\leqslant O(1)\exp(-\delta_7\varepsilon M)\end{aligned}$$

which implies the desired result. ∎

By Lemma 3.5 we then know that $\mathrm{Var}(Z_{i,n})$ has a uniform lower bound. We are now in a position to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. Let

$$\phi(i)=(\tilde{E}\log\phi_1(i)/\phi_2(i))^2$$

By definition we know that

$$|\phi_1(i)/\phi_2(i)|\leqslant(\tilde{E}\log\exp(2\beta M))^2=(2\beta M)^2$$

By Hölder's inequality and Lemma 3.5 we have

$$(\tilde{E}I_{\{\phi_1(i)\leqslant\exp(\delta_1 M)\phi_2(i)\}}\log\phi_1(i)/\phi_2(i))^2\leqslant O(1)\exp(-\delta_2 M/4)$$

which implies the following, provided $M\geqslant 1$ is large enough:

$$\begin{aligned}\phi(i)&\geqslant -C_3\exp(-\delta_2 M/4)\\&\qquad+\tfrac{1}{2}[\tilde{E}I_{\{\phi_1(i)\geqslant\exp(\delta_1 M)\phi_2(i)\}}\log(\phi_1(i)/\phi_2(i))]^2\\&\geqslant O(1)M-C_3\exp(-\delta_2 M/4)\geqslant C_4\end{aligned}$$

for some constants $C_3, C_4 \in (0, \infty)$. By (3.1) and Lemma 3.2 we know that if $M \geqslant 1$ is large enough

$$\begin{aligned}\operatorname{Var}(\log Z^{(0,n)}) &\geqslant \sum_{i=1}^{l_n} \operatorname{Var}(Z_{i,n}) \\ &\geqslant \frac{1}{2} \sum_{i=1}^{l_n} E' I_{\{a_j = t_j,\, a'_j = t'_j,\, (i-1)M+1 \leqslant j \leqslant iM\}}\, \phi(i) \\ &\geqslant \frac{1}{2} 2^{-M} 2^{-M} \sum_{i=1}^{l_n} \phi(i) \geqslant O(1)n\end{aligned}$$

which proves the lower bound.

Now it remains to show (by Lemma 3.2)

$$\operatorname{Var}(Y_{i,n}) \leqslant O(1), \qquad i = 1, \dots, l_n \tag{3.5}$$

Without loss of generality, we may prove (3.5) only for $i = 1$. By the definition of $Y_{1,n}$ we have

$$Y_{1,n} = E' \log E \exp\left(\beta \sum_{i=1}^{M} a_i U(S_i) + \beta \sum_{i=M+1}^{n} b_i U(S_i)\right)$$

In this case we have

$$\begin{aligned}\exp(-|\beta|\, M) E \exp\left(\beta \sum_{i=M+1}^{n} b_i U(S_i)\right) &\leqslant E \exp\left(\beta \sum_{i=1}^{M} u_i U(S_i) + \beta \sum_{i=M+1}^{n} b_i U(S_i)\right) \\ &\leqslant \exp(|\beta|\, M) E \exp\left(\beta \sum_{i=M+1}^{n} b_i U(S_i)\right)\end{aligned}$$

for any given $u_1, \dots, u_M \in \{-1, 1\}$. Hence,

$$\left| Y_{1,n} - \log E \exp\left(\beta \sum_{i=M+1}^{n} b_i U(S_i)\right)\right| \leqslant |\beta|\, M$$

which implies

$$\tilde{E}\, |\tilde{E} Y_{1,n} - Y_{1,n}|^2 \leqslant O(1)\, |\beta|^2 M^2$$

This then completes the proof of (3.5). ∎

4. CENTRAL LIMIT THEOREM

The main aim of this section is to prove that the central limit theorem holds for the free energy (see Theorem 4.4 below). To get this limit theorem, we will first derive a reasonable estimate on the fourth moment of the free energy, and then prove that the free energy can be expressed asymptotically as the sum of some independent identically distributed random variables. Using the estimate on the fourth moment of the free energy, we can show that the Lindeberg condition is fulfilled for this system, and then get the central limit theorem. To this end, let us first prove two lemmas. Lemma 4.1 below concerns the behavior of the variance of the free energy. Proposition 3.1 given before will be used in the proof of Lemma 4.1. Lemma 4.2 below will be used to verify that the Lindeberg condition is satisfied for this system.

Lemma 4.1. For any given $\beta \neq 0$ there is a constant $\gamma(\beta) \in (0, \infty)$ such that

$$\lim_{n \to \infty} \frac{\mathrm{Var}(\log Z^{(0,n)})}{n} = \gamma(\beta)$$

where $\mathrm{Var}(\log Z^{(0,n)}) = \tilde{E}(\log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)})^2$.

The main idea to prove this lemma is the same as the one given in the proof of Lemma 2.4, where $\lim_{n \to \infty}(R_n/n)$ has been proven to exist. We will prove that $\mathrm{Var}(\log Z^{(0,n)})$ has asymptotically a subadditive property, which implies the desired conclusion.

Proof. Let

$$\xi_n = \log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)}$$

and $r_n = \tilde{E}\xi_n^2$. For any given $\alpha \in (0, 1)$ we let m satisfy $n^\alpha \leqslant m \leqslant n$. From the proof of Lemma 2.3 we see that there is a constant $K_0 \geqslant 1$ such that

$$\tilde{P}\left(Z^{(0,n+m)} \geqslant n^{K_0} Z^{(0,n)} E \exp\left(\beta \sum_{i=1}^{m} b_{i+n} U(S_i)\right)\right) \leqslant O(1) n^{-4}$$

In other words, we have

$$\tilde{P}\left(\log Z^{(0,n+m)} \geqslant \log n^{K_0} Z^{(0,n)} + \log E \exp\left(\beta \sum_{i=1}^{m} b_{i+n} U(S_i)\right)\right) \leqslant O(1)\, n^{-4} \tag{4.1}$$

By (2.13) we know that there is a constant $K_1 \in (0, \infty)$ such that

$$\tilde{P}\left(E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) I_{\{S_n \neq 0\}} \geqslant n^{K_1} \tilde{Z}^{(0,n)}\right) \leqslant O(1)\, n^{-4}$$

where $\tilde{Z}^{(0,n)}$ was defined at the beginning of Section 2. Using this, we show that

$$\begin{aligned}\tilde{E} \log Z^{(0,n)} &\leqslant \tilde{E} \log((n^{K_1}+1)\, \tilde{Z}^{(0,n)}) \\ &\quad + O(1)\, n\tilde{P}\left(E \exp\left(\beta \sum_{i=1}^{n} b_i U(S_i)\right) I_{\{S_n \neq 0\}} \geqslant n^{K_1} \tilde{Z}^{(0,n)}\right) \\ &\leqslant O(1) \log n + \tilde{E} \log \tilde{Z}^{(0,n)}\end{aligned}$$

which implies

$$\begin{aligned}\tilde{E} \log Z^{(0,n+m)} &\geqslant \tilde{E} \log \tilde{Z}^{(0,n)} + \tilde{E} \log E \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right) \\ &\geqslant -C_1 \log n + \tilde{E} \log Z^{(0,n)} + \tilde{E} \log Z^{(0,m)}\end{aligned}$$

for some constant $C_1 \in (0, \infty)$. Let

$$\xi'_m(n) = \log E \exp\left(\beta \sum_{i=1}^{m} b_{n+i} U(S_i)\right) - \tilde{E} \log E \exp\left(\beta \sum_{i=1}^{m} b_{b+i} U(S_i)\right)$$

By (4.1) we have

$$\tilde{P}(\xi_{n+m} \geqslant K_0 \log n + C_1 \log n + \xi_n + \xi'_m(n)) \leqslant O(1)\, n^{-4}$$

Similarly, we prove that

$$\tilde{P}(\xi_{n+m} \leqslant -K'_0 \log n - C'_1 \log n + \xi_n + \xi'_m(n)) \leqslant O(1)\, n^{-4}$$

for some constants K'_0, $C'_1 \in (0, \infty)$. It is clear that $\xi_n \leqslant |\beta|\, n$ and $\tilde{E}\xi_n = \tilde{E}\xi'_m(n) = 0$. Since ξ_n and $\xi'_m(n)$ are independent, we get

$$\tilde{E}\xi^2_{n+m} \leqslant O(1)(\log n)^2 + O(1) \log n(\tilde{E}\, |\xi_n| + \tilde{E}\, |\xi'_m(n)|) + \tilde{E}\xi^2_n + \tilde{E}(\xi'_m(n))^2$$

By Proposition 3.1 we know that

$$\tilde{E}\xi^2_{n+m} \leqslant O(1)\, n^{1/2} \log n + \tilde{E}\xi^2_n + \tilde{E}(\xi'_m(n))^2$$

Thus there is a constant $C_2 \in (0, \infty)$ such that

$$r_{n+m} \leqslant C_2 n^{1/2} \log n + r_n + r_m \tag{4.2}$$

We now use the approach given in the proof of Lemma 2.4 to prove that the following limit exists:

$$\lim_{n \to \infty} \frac{r_n}{n} = \gamma(\beta) \tag{4.3}$$

Let

$$\tilde{r}_n = C_2 n^{1/2} \log n + r_n$$

One can easily show that for any given $\delta \in (0, 1)$ there is a large constant $m_0 \geqslant 1$ such that

$$(n+m)^{1/2} \log(n+m) \leqslant n^{1/2} \log n + m^{1/2} \log m$$

if $n^\delta \vee m_0 \leqslant m \leqslant n$ and $m \geqslant m_0$. Using this, we get

$$\tilde{r}_{n+m} \leqslant \tilde{r}_n + \tilde{r}_m$$

if $n^\delta \vee m_0 \leqslant m \leqslant n$. Thus, by a similar argument as in the proof of Lemma 2.4 we show that (4.3) is indeed true. By Proposition 3.1 we know that $\gamma(\beta)$ is finite and positive. ∎

As mentioned before, we will prove that the Lindeberg condition is fulfilled for the present system. To do so, we first derive a reasonable estimate on the fourth moment of the free energy.

Lemma 4.2. There is a constant $C_5 \in (0, \infty)$ such that

$$\tilde{E}(\log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)})^4 \leqslant C_5 n^2 \tag{4.4}$$

It is not easy to give a direct estimate for the fourth moment of the centered free energy. The proof of Lemma 4.2 given below will be divided into two steps. First we will prove that the desired estimate holds for $n = 2^m$. For this purpose we set

$$\alpha_m = 2^{-2m} \tilde{E}(\log Z^{(0,2^m)} - \tilde{E} \log Z^{(0,2^m)})^4$$

We will derive an estimate of α_m in terms of α_{m-1} and then use this estimate to prove the boundedness of the sequence $\{\alpha_m\}$. Second we shall give an estimate for the fourth moment of the centered free energy in terms of the sequence $\{\alpha_m\}$. To do so we need some further considerations.

Proof. From the argument given before we see that there is a constant $C_6 \in [1, \infty)$ such that

$$2\tilde{E} \log Z^{(0,2^{m-1})} - C_6^{-1} m \leqslant \tilde{E} \log Z^{(0,2^m)} \leqslant C_6 m + 2\tilde{E} \log Z^{(0,2^{m-1})}$$

and

$$\tilde{P}\left(\log Z^{(0,2^m)} \leqslant \log Z^{(0,2^{m-1})} + \log E \exp\left(\beta \sum_{i=1}^{2^{m-1}} b_{i+2^{m-1}} U(S_i)\right) - C_6^{-1} m\right)$$

$$+ \tilde{P}\left(\log Z^{(0,2^m)} \geqslant \log Z^{(0,2^{m-1})} + \log E \exp\left(\beta \sum_{i=1}^{2^{m-1}} b_{i+2^{m-1}} U(S_i)\right) + C_6^{-1} m\right)$$

$$\leqslant O(1)\, 2^{-4m}$$

Let

$$\eta_{m-1} = \log Z^{(0,2^{m-1})} + \log E \exp\left(\beta \sum_{i=1}^{2^{m-1}} b_{i+2^{m-1}} U(S_i)\right) - 2\tilde{E} \log Z^{(0,2^{m-1})}$$

Then we show that

$$\begin{aligned}
&\tilde{E}(\log Z^{(0,2^m)} - \tilde{E} \log Z^{(0,2^m)})^4 \\
&\quad \leqslant O(1) + \tilde{E}(|\eta_{m-1}| + O(1)m)^4 \\
&\quad \leqslant \tilde{E}\, |\eta_{m-1}|^4 + O(1) \sum_{i=1}^{4} m^i \tilde{E}\, |\eta_{m-1}|^{4-i} \\
&\quad \leqslant \tilde{E}\, |\eta_{m-1}|^4 + O(1) \sum_{i=1}^{4} m^i (\tilde{E}\, |\eta_{m-1}|^4)^{(4-i)/4}
\end{aligned} \tag{4.5}$$

Since $\log Z^{(0,2^{m-1})}$ and $\log E \exp(\beta \sum_{i=1}^{2^{m-1}} b_{i+2^{m-1}} U(S_i))$ are independent and have the same distribution, we have

$$\begin{aligned}
\tilde{E}\, |\eta_{m-1}|^4 &\leqslant 2\tilde{E}\, |\log Z^{(0,2^{m-1})} - \tilde{E} \log Z^{(0,2^{m-1})}|^4 \\
&\quad + O(1)(\tilde{E}\, |\log Z^{(0,2^{m-1})} - \tilde{E} \log Z^{(0,2^{m-1})}|^2)^2 \\
&\leqslant 2 2^{2(m-1)} \alpha_{m-1} + O(1)\, 2^{2(m-1)}
\end{aligned}$$

Hence, by (4.5) we can show that there are constants $\varepsilon \in (0, 1/2)$, $m_0 \geqslant 1$, and $C_7 \in (0, \infty)$ such that

$$\alpha_m \leqslant (\tfrac{1}{2} + \varepsilon)\, \alpha_{m-1} + C_7, \qquad \forall m \geqslant m_0$$

By this we know that for some constant $C_8 \in (0, \infty)$

$$\alpha_m \leqslant C_8, \qquad \forall m \geqslant 1$$

In other words, (4.4) holds for $n = 2^m$, $\forall m \geqslant 1$.

We now prove that (4.4) holds for all $n \geqslant 1$. As in the proof of Lemma 2.4, we show that for any $n \geqslant 1$ there are $l_0,..., l_k \in \{0, 1\}$ such that $l_k = 1$ and

$$n = \sum_{i=0}^{k} l_i 2^i$$

Let $u = \max\{i \leqslant k-1:\ l_i = 1\}$ and $m_u = \sum_{i=0}^{u} l_i 2^i$. It is clear that $n = 2^k + m_u$. If $u \leqslant k/2$, then

$$\begin{aligned}
&\tilde{E}(\log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)})^4 \\
&\quad \leqslant O(1) \tilde{E}\left(\log E \exp\left(\beta \sum_{i=1}^{2^k} b_{i+m_u} U(S_i)\right) - \tilde{E} \log Z^{(0,2^k)}\right)^4 \\
&\qquad + O(1)(\log n)^4 + O(1)\, \tilde{E}(\log Z^{(0,m_u)})^4 \\
&\quad \leqslant O(1)\, \tilde{E}(\log Z^{(0,2^k)} - \tilde{E} \log Z^{(0,2^k)})^4 + O(1)((\log n)^4 + m_u^4) \\
&\quad \leqslant O(1)(2^{2k} + (\log n)^4 + 2^{2k}) \\
&\quad \leqslant O(1)\, n^2
\end{aligned}$$

Hence, (4.4) holds in this case.

We now assume $u \geqslant k/2$ and prove the desired result [i.e. (4.4)] by induction on k. Clearly,

$$n^{1/4} \leqslant m_u \leqslant n$$

if $n \geqslant 1$ is large enough. Let

$$\begin{aligned}
&\zeta_1(n) = \log Z^{(0,m_u)} - \tilde{E} \log Z^{(0,m_u)} \\
&\zeta_2(n) = \log E \exp\left(\beta \sum_{i=1}^{2^k} b_{i+m_u} U(S_i)\right) - \tilde{E} \log E \exp\left(\beta \sum_{i=1}^{2^k} b_{i+m_u} U(S_i)\right)
\end{aligned}$$

As an induction assumption, we may assume that

$$\tilde{E}\, |\zeta_1(n)|^4 \leqslant K \cdot 2^{2u}$$

for some constant $K \geqslant 1$ whose value will be specified below. By a similar argument as in the derivation of (4.5) we can show that

$$\begin{aligned}
&\tilde{E}\, |\log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)}|^4 \\
&\quad \leqslant \tilde{E}\, |\zeta_1(n) + \zeta_2(n)|^4 + O(1) \sum_{i=1}^{4} (\log n)^i (\tilde{E}\, |\zeta_1(n) + \zeta_2(n)|^4)^{(4-i)/4} \qquad (4.6)
\end{aligned}$$

If $\tilde{E}\,|\zeta_1(n)|^4 \leqslant M\tilde{E}\,|\zeta_2(n)|^4$ for some $M \geqslant 1$, then

$$\text{l.h.s. of (4.6)} \leqslant O(1)\,\tilde{E}\,|\zeta_2(n)|^4 \leqslant O(1)\,n^2$$

We now assume $\tilde{E}\,|\zeta_1(n)|^4 \geqslant M\tilde{E}\,|\zeta_2(n)|^4$, where $M \geqslant 1$ will be specified below. Then

$$\begin{aligned}\tilde{E}\,|\zeta_1(n)+\zeta_2(n)|^4 &= \tilde{E}\,|\zeta_1(n)|^4 + \tilde{E}\,|\zeta_2(n)|^4 + 6\tilde{E}\,|\zeta_1(n)|^2\,\tilde{E}\,|\zeta_2(n)|^2 \\ &\leqslant \left(1+\frac{1+6M^{1/2}}{M}\right)\tilde{E}\,|\zeta_1(n)|^4\end{aligned}$$

Therefore, for any given $\varepsilon \in (0,1)$ there is $M_0 \geqslant 1$ such that

$$\text{l.h.s. of (4.6)} \leqslant (1+\varepsilon)\tilde{E}\,|\zeta_1(n)|^4$$

if $n \geqslant 1$ is large enough and $\tilde{E}\,|\zeta_1(n)|^4 \geqslant M_0\tilde{E}\,|\zeta_2(n)|^4$. Using this, we can show that there is a constant $C_9 \in (0,\infty)$ such that l.h.s. of (4.6) is less than

$$\begin{aligned}(1+\varepsilon)\tilde{E}\,|\zeta_1(n)|^4 + C_9 n^2 &\leqslant (1+\varepsilon)K\cdot 2^{2u} + C_9 n^2 \\ &\leqslant \tfrac{1}{2}K\cdot 2^{2k} + 4C_9 2^{2k}\end{aligned}$$

By letting $K = 8C_9$ we then obtain that

$$\tilde{E}\,|\log Z^{(0,2^k+m_u)} - \tilde{E}\log Z^{(0,2^k+m_u)}|^4 \leqslant K\cdot 2^{2k}$$

which proves the desired result. ∎

As a corollary to Lemma 4.2 we get a strong law of large numbers for $\log Z^{(0,n)}$.

Corollary 4.3. The following strong law of large numbers holds:

$$\lim_{n\to\infty}\frac{\log Z^{(0,n)}}{\nu(\beta)n} = 1 \qquad \text{a.e.-}\tilde{P}$$

Proof. By Lemma 4.2 we know that

$$\tilde{E}\left|\frac{\log Z^{(0,n)}}{\tilde{E}\log Z^{(0,n)}} - 1\right|^4 \leqslant O(1)\,n^{-2}$$

By the Borel–Cantelli lemma we see that

$$\lim_{n\to\infty}\left|\frac{\log Z^{(0,n)}}{\tilde{E}\log Z^{(0,n)}} - 1\right| = 0 \qquad \text{a.e.-}\tilde{P}$$

Thus, we get the desired result from Proposition 2.6. ∎

Having Lemmas 4.1 and 4.2, we can now prove the central limit theorem for the free energy.

Theorem 4.4. For any given $\beta \neq 0$ we have the following results:

$$\frac{\log Z^{(0,n)} - v(\beta)n}{\gamma(\beta)^{1/2} n^{1/2}} \xrightarrow{\mathscr{D}} N(0, 1), \qquad n \to \infty$$

$$\frac{\log \tilde{Z}^{(0,2n)} - 2v(\beta)n}{\gamma(\beta)^{1/2}(2n)^{1/2}} \xrightarrow{\mathscr{D}} N(0, 1), \qquad n \to \infty$$

where $\mathscr{D}$ represents the weak convergence, $N(0, 1)$ is a random variable with the standard normal distribution, and $v(\beta)$ and $\gamma(\beta)$ were determined respectively in Lemmas 2.4 and 4.1.

Proof. Let $\alpha_0 \in (1/2, 1)$. Without loss of generality, we may assume $n^{\alpha_0} = [n^{\alpha_0}]$ is even, and $n = u_n n^{\alpha_0}$ for some integer $u_n \geqslant 1$. Let

$$\theta_i = E \exp\left(\beta \sum_{j=1}^{n^{\alpha_0}} b_{(i-1)n^{\alpha_0}+j} U(S_j)\right) - \tilde{E} \log Z^{(0,n^{\alpha_0})}$$

Clearly, $\theta_1,..., \theta_{u_n}$ are i.i.d., and $u_n = O(n^{1-\alpha_0})$. By Lemma 4.1 we know that

$$\tilde{E}\theta_i^2 = \gamma(\beta)\, n^{\alpha_0}(1 + o(1)), \qquad n \to \infty$$

and so

$$\operatorname{Var}\left(\sum_{i=1}^{u_n} \theta_i\right) = \gamma(\beta)\, n(1 + o(1)), \qquad n \to \infty$$

By Lemma 4.2 we know that the following holds for any given $\varepsilon \in (0, 1)$:

$$\begin{aligned}\frac{1}{\gamma(\beta)n} \sum_{i=1}^{u_n} \tilde{E}\theta_i^2 I_{\{|\theta_i| \geqslant \varepsilon(\gamma(\beta)n)^{1/2}\}} &\leqslant \frac{1}{\gamma(\beta)n} \frac{1}{\varepsilon^2\gamma(\beta)n} \sum_{i=1}^{u_n} \tilde{E}\theta_i^4 \\ &\leqslant O(1)\, n^{-2} \sum_{i=1}^{u_n} n^{2\alpha_0} \\ &\leqslant O(1)\, u_n^{-1}\end{aligned}$$

which goes to zero as $n \to \infty$ (the above property is called Lindeberg's condition). Hence, the following central limit theorem holds (see the proof of ref. 7, Theorem 4.5):

$$\frac{\sum_{i=1}^{u_n} \theta_i}{\operatorname{Var}^{1/2}(\sum_{i=1}^{u_n} \theta_i)} \xrightarrow{\mathscr{D}} N(0, 1), \qquad n \to \infty$$

Let

$$\tilde{\theta}_i = \log Z_{0,0}^{((i-1)n^{\alpha_0}, in^{\alpha_0})} - \tilde{E} \log Z_{0,0}^{((i-1)n^{\alpha_0}, in^{\alpha_0})}$$

By (2.13) we have

$$\tilde{E}\, |\theta_i - \tilde{\theta}_i|^2 \leqslant O(1)(\log n)^2$$

and so

$$\tilde{E} \left| \sum_{i=1}^{u_n} (\theta_i - \tilde{\theta}_i) \right|^2 \leqslant O(1)(\log n)^2 u_n^2$$

As in the proof of Lemma 2.2 we can also show that there is a constant $K_2 \geqslant 1$ such that

$$\tilde{P}(\theta_i' \geqslant n^{K_2} Z_{0,0}^{((i-1)n^{\alpha_0}, in^{\alpha_0})}) \leqslant O(1)\, n^{-4}$$

where

$$\theta_i' = \max_{x \in Z^1} E_x \exp\left(\beta \sum_{j=1}^{n^{\alpha_0}} b_{(i-1)n^{\alpha_0}+j} U(S_j)\right)$$

By definition we know that

$$\prod_{i=1}^{u_n} Z_{0,0}^{((i-1)n^{\alpha_0}, in^{\alpha_0})} \leqslant Z^{(0,n)} \leqslant \prod_{i=1}^{u_n} \theta_i'$$

Then we have

$$\tilde{E} \left| \log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)} - \sum_{i=1}^{u_n} \tilde{\theta}_i \right|^2 \leqslant O(1)(\log n)^2 u_n^2$$

Therefore,

$$\tilde{E} \left| \log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)} - \sum_{i=1}^{u_n} \theta_i \right|^2$$
$$\leqslant O(1)(\log n)^2\, u_n^2 \leqslant O(1)\, n^{2(1-\alpha_0)}(\log n)^2$$

Using this, we get

$$\frac{\log Z^{(0,n)} - \tilde{E} \log Z^{(0,n)}}{(\gamma(\beta) n)^{1/2}} \xrightarrow{\mathscr{D}} N(0, 1), \qquad n \to \infty$$

By Proposition 2.6 we know that

$$|\tilde{E} \log Z^{(0,n)} - \nu(\beta) n| \leqslant O(1) \log n$$

From this we easily get

$$\frac{\log Z^{(0,n)} - \nu(\beta) n}{\gamma(\beta)^{1/2} n^{1/2}} \xrightarrow{\mathscr{L}} N(0, 1), \qquad n \to \infty$$

Similarly, we can get the other weak convergence asserted in this theorem, and the proof of Theorem 4.4 is then complete. ▮

As a corollary to Theorem 4.4 and Proposition 2.6, we have the following estimate on the partition function.

Corollary 4.5. For any given $\beta \neq 0$ and $x_1 < x_2$, we have

$$\lim_{n \to \infty} \tilde{P}(\exp(x_1 \gamma(\beta)^{1/2} n^{1/2}) \leqslant Z^{(0,n)} e^{-\nu(\beta) n} \leqslant \exp(x_2 \gamma(\beta)^{1/2} n^{1/2}))$$

$$= (2\pi)^{-1/2} \int_{x_1}^{x_2} \exp\left(-\frac{|y|^2}{2}\right) dy$$

5. RECURRENCE

From now on we discuss the sample path properties of the random walk $\{S_i\}$ under the probability measure $P^{(0,n)}$. In this section we first consider a property which is related to recurrence. Let $\tau_0(n) = 0$,

$$\tau_i(n) = \inf\{m > \tau_{i-1}(n) \colon S_m = 0\} \wedge n, \qquad i = 1, 2, \ldots$$

τ_i, $i = 0, 1, 2, \ldots$, are stopping times. It is easy to show that $\Delta_n = \max_{1 \leqslant i < \infty}(\tau_i(n) - \tau_{i-1}(n))$, where Δ_n was defined in Section 1. Before stating the main result in this section, let us first prove the existence of an exponent $\mu(\beta)$ describing the behavior of Δ_n (see Lemma 5.2 and Theorem 5.3 below). For this purpose we prove two lemmas.

Lemma 5.1. For any given $\beta \neq 0$ and $\varepsilon \in (0, 1)$ the following is always true:

$$\tilde{Z}^{(0,2n)} \geqslant e^{-\varepsilon n} \exp\left(|\beta| \left|\sum_{i=1}^{2n} b_i\right|\right)$$

if $n \geqslant 1$ is large enough, where $\tilde{Z}^{(0,2n)}$ was defined at the beginning of Section 2.

Although this lemma is simple to prove, it already implies that the exponent $\mu(\beta)$ given in the next lemma is positive.

Proof. For any given $M \geqslant 1$, by (2.1) and (2.3) we know that there is a constant $C_1 \in (0, \infty)$, which is independent of $M \geqslant 1$, such that

$$\tilde{Z}^{(0,nM)} \geqslant \prod_{i=1}^{n} \left(C_1 M^{-2} \exp\left(\left| \beta \sum_{i=1}^{M} b_i \right| \right) \right)$$
$$\geqslant ((C_1 M^{-2})^{1/M})^{nM} \exp\left(\left| \beta \sum_{i=1}^{nM} b_i \right| \right)$$

Thus we get the desired result by letting $M \geqslant 1$ be large enough [e.g., $(C_1 M^{-2})^{1/M} \geqslant e^{-\varepsilon}$]. ∎

Lemma 5.2. Let

$$Y_n(j) = \log \tilde{E}\left(E \exp\left(\beta \sum_{i=1}^{2n-1} b_i(U(S_i) + j) \right) I_{\{S_{2n}=0\}} \right)^{-1}, \qquad j = 1, -1$$

$$Y'_n(j) = \log \tilde{E}\left(E \exp\left(\beta \sum_{i=1}^{n} b_i(U(S_i) + j) \right) \right)^{-1}, \qquad j = 1, -1$$

For any given $\beta \neq 0$ there is a constant $\mu(\beta) \in (0, \infty)$ such that

$$\lim_{n \to \infty} (2n)^{-1}\, Y_n(j) = \lim_{n \to \infty} n^{-1} Y'_n(j) = -\mu(\beta), \qquad j = -1, 1$$

As in the proof of Lemma 2.3, it is easy to show that Y_n has asymptotically a subadditive property. Using this property, one can easily prove the existence of the exponent $\mu(\beta)$. By Theorem 4.4 we can then easily deduce that the exponent $\mu(\beta)$ is finite. By Lemma 5.1 we then succeed in proving that $\mu(\beta)$ is also positive.

Proof. By the symmetry properties of $\{b_i\}_{i \geqslant 1}$ and $\{S_i\}_{i \geqslant 1}$ we know that $Y_n(1) = Y_n(-1)$. By considering the special paths with $\{S_{2m} = 0\}$ and the Markov property we have that for any given $n, m \geqslant 1$

$$Y_{n+m}(1) \leqslant \log \tilde{E}\left(E \exp\left(\beta \sum_{i=1}^{2n} b_i(U(S_i) + 1) \right) I_{\{S_{2n}=0\}} \right)^{-1}$$
$$\times \left(E \exp\left(\beta \sum_{i=1}^{2m-1} b_{2n+i}(U(S_i) + 1) \right) I_{\{S_{2m}=0\}} \right)^{-1}$$
$$\leqslant \log \tilde{E}\left(E \exp\left(\beta \sum_{i=1}^{2n-1} b_i(U(S_i) + 1) \right) I_{\{S_{2n}=0\}} \right)^{-1} + \log \tilde{E} e^{-\beta b_{2n}}$$
$$+ \log \tilde{E}\left(E \exp\left(\beta \sum_{i=1}^{2m-1} b_{2n+i}(U(S_i) + 1) \right) I_{\{S_{2m}=0\}} \right)^{-1}$$
$$\leqslant Y_n(1) + Y_m(1) + \tfrac{1}{2}(e^{-\beta} + e^{\beta})$$

As in the proof of Lemma 2.4, we can show that there is a constant $\mu(\beta)$ such that

$$\lim_{n \to \infty} (2n)^{-1} Y_n(1) = -\mu(\beta)$$

We now prove $\mu(\beta) \in (0, \infty)$. Indeed, by Theorem 4.4 we have that the following also holds:

$$\frac{\log E \exp(\beta \sum_{i=1}^{2n-1} b_i U(S_i)) I_{\{S_{2n}=0\}} - 2\nu(\beta) n}{\gamma(\beta)^{1/2} (2n)^{1/2}} \xrightarrow{\mathscr{D}} N(0, 1), \qquad n \to \infty$$

Hence, there are a constant $M_1 \geqslant 1$ and a subset $\tilde{\Omega}_1 \subset \tilde{\Omega}$ with $\tilde{P}(\tilde{\Omega}_1) \geqslant 3/4$ such that

$$E \exp\left(\beta \sum_{i=1}^{2n-1} b_i U(S_i)\right) I_{\{S_{2n}=0\}} \leqslant \exp[2n\nu(\beta) + M_1 \gamma(\beta)^{1/2} n^{1/2}]$$

if $\tilde{\Omega}_1$ has occurred. We also know that there are a constant $M_2 \geqslant 1$ and a subset $\tilde{\Omega}_2 \subset \tilde{\Omega}$ with $\tilde{P}(\tilde{\Omega}_2) \geqslant 3/4$ such that

$$\exp\left(\beta \sum_{i=1}^{n} b_i\right) \leqslant \exp(M_2 \, |\beta| \, n^{1/2})$$

if $\tilde{\Omega}_2$ has occurred. Using these inequalities, we get

$$\begin{aligned} Y_n(1) &\geqslant \log \tilde{E} I_{\tilde{\Omega}_1 \cap \tilde{\Omega}_2} \exp[-2\nu(\beta) n - M_1 \gamma(\beta)^{1/2} n^{1/2} - M_2 \, |\beta| \, n^{1/2}] \\ &\geqslant -2\nu(\beta) n - M_1 \nu(\beta)^{1/2} n^{1/2} - M_2 \, |\beta| \, n^{1/2} - \log 2 \end{aligned}$$

which implies

$$\mu(\beta) \leqslant \nu(\beta) < \infty, \qquad \forall \beta \in R^1 \backslash \{0\}$$

On the other hand, by Lemma 2.1 we know that there are constants $\delta_1, \delta_2 > 0$ such that

$$\tilde{P}(\tilde{Z}^{(0,2n)} \leqslant \exp(\delta_1 n)) \leqslant O(1) \exp(-\delta_2 n) \tag{5.1}$$

By a large-deviation result we know that there is a constant $\delta_3 \in (0, \infty)$ such that

$$\tilde{P}\left(\left|\beta \sum_{i=1}^{2n} b_i\right| \geqslant \delta_1 n/2\right) \leqslant O(1) \exp(-\delta_3 n)$$

Thus, by Lemma 5.1 and (5.1) we obtain

$$\begin{aligned}
&\tilde{E}\left(E\exp\left(\beta\sum_{i=1}^{2n-1} b_i(U(S_i)+1)\right)I_{\{S_{2n}=0\}}\right)^{-1}\\
&\leqslant 2\tilde{E}\left(\tilde{Z}^{(0,2n)}\exp\left(\beta\sum_{i=1}^{2n} b_i\right)\right)^{-1}\\
&\leqslant 2\tilde{E}I_{\{\tilde{Z}^{(0,2n)}\leqslant\exp(\delta_1 n)\}}\left(\tilde{Z}^{(0,2n)}\exp\left(\beta\sum_{i=1}^{2n} b_i\right)\right)^{-1}\\
&\quad+2\tilde{P}\left(\tilde{Z}^{(0,2n)}\geqslant\exp(\delta_1 n),\ \exp\beta\sum_{i=1}^{2n} b_i\right)\leqslant\exp(\delta_1 n/2)\Big)\exp(-\delta_1 n/2)\\
&\quad+2\tilde{E}I_{\{\exp(\beta\sum_{i=1}^{2n} b_i)\geqslant\exp(\delta_1 n/2)\}}\left(\tilde{Z}^{(0,2n)}\exp\left(\beta\sum_{i=1}^{2n} b_i\right)\right)^{-1}\\
&\leqslant O(1)\exp(\varepsilon n-\delta_2 n)+2\exp(-\tfrac{1}{2}\delta_1 n)\\
&\quad+O(1)\exp(\varepsilon n)\tilde{P}\left(\left|\beta\sum_{i=1}^{2n} b_i\right|\geqslant\delta_1 n/2\right)\\
&\leqslant O(1)\exp(\varepsilon n)\exp(-\delta_2\wedge\delta_3 n)+2\exp(-\delta_1 n/2)\\
&\leqslant O(1)\exp(-\tfrac{1}{2}\delta_1\wedge\delta_2\wedge\delta_3 n)
\end{aligned}$$

if $\varepsilon\in(0,\frac{1}{2}\delta_2\wedge\delta_3)$ and $n\geqslant 1$ is large enough. Using this, we obtain that

$$Y_n(1)\leqslant -\tfrac{1}{2}\delta_1\wedge\delta_2\wedge\delta_3 n$$

which leads to $\mu(\beta)\geqslant\frac{1}{2}\delta_1\wedge\delta_2\wedge\delta_3>0$.

It is clear that $Y_n(1)\geqslant Y'_{2n-1}(1)$. By Lemma 5.1 we can show that for any given $\varepsilon\in(0,1)$ there is a constant $C_2\in(0,\infty)$ such that

$$Y'_{2n-1}(1)\geqslant\log C_2-\varepsilon n+Y_n(1)$$

if $n\geqslant 1$ is large enough. Since $\varepsilon>0$ is arbitrary, we have

$$\lim_{n\to\infty}(n)^{-1}\,Y'_{2n-1}(1)=\lim_{n\to\infty}(2n)^{-1}\,Y_n(1)$$

On the other hand, it is easy to show that

$$\lim_{n\to\infty}n^{-1}\,Y'_n(1)=\lim_{n\to\infty}(2n)^{-1}\,Y'_{2n-1}(1),$$

which proves the desired result. ∎

Remark. We already proved that

$$\mu(\beta) \leqslant \nu(\beta), \qquad \forall \beta \neq 0$$

It would also be very interesting to give an expression for $\mu(\beta)$ in terms of β.

Our main theorem concerning the behavior of Δ_n as $n \to \infty$ is as follows.

Theorem 5.3. For any given $\beta \neq 0$ and $\varepsilon \in (0, \mu(\beta))$, we have

$$P^{(0,n)}((\log n)^{-1} \Delta_n \in [(\mu(\beta)+\varepsilon)^{-1}, (\mu(\beta)-\varepsilon)^{-1}]) \xrightarrow{\tilde{P}} 1, \qquad n \to \infty$$

where $\mu(\beta)$ was defined in Lemma 5.2.

The proof of Theorem 5.3 is very involved. We will divide it into two steps. First we will prove

$$P^{(0,n)}((\log n)^{-1} \Delta_n \leqslant (\mu(\beta)-\varepsilon)^{-1}) \xrightarrow{\tilde{P}} 1, \qquad n \to \infty$$

This part can be easily proven by using the definition of $\mu(\beta)$ given in Lemma 5.2. Lemma 5.4 below will deal with this part. Unfortunately, the other part of the proof of Theorem 5.3 is not so easy; in fact a lot of computation will be involved in the proof of this part (see Lemmas 5.5 and 5.6 below).

Let $\sigma_i = \inf\{m > i: S_m = 0\} \wedge n - i$. Then σ_i is a stopping time.

Lemma 5.4. For any given $\beta \neq 0$ and $\varepsilon \in (0, \mu(\beta))$,

$$\begin{aligned} &\tilde{E}P^{(0,n)}(S_i = 0, \sigma_i \geqslant (\mu(\beta)-\varepsilon)^{-1} \log n) \\ &\quad \leqslant O(1)\, n^{-1-(\varepsilon/2)(\mu(\beta)-\varepsilon)^{-1}}, \qquad i = 1, \dots, n, \quad \forall n \geqslant 1 \end{aligned} \tag{5.2}$$

Proof. It suffices to prove the above estimate for $i \leqslant n - [(\mu(\beta)-\varepsilon)^{-1} \log n]$. We remark that if $\sigma_i = j$, then $U(S_1) = \cdots = U(S_{j-1}) \neq 0$. Using again the Markov property, we can show that

$$\begin{aligned} &EI_{\{S_i=0,\, \sigma_i \geqslant (\mu(\beta)-\varepsilon)^{-1} \log n\}} \exp\left(\beta \sum_{j=1}^{n} b_j U(S_j)\right) \\ &\leqslant \sum_{j=[(\mu(\beta)-\varepsilon)^{-1} \log n]}^{n-i} E \exp\left(\beta \sum_{k=1}^{i} b_k U(S_k)\right) I_{\{S_i=0\}} \\ &\quad \times E \exp\left(\beta \sum_{l=1}^{j-1} b_{l+j} U(S_1)\right) I_{\{S_j=0,\, U(S_1)=\cdots=U(S_{j-1})\neq 0\}} \\ &\quad \times E \exp\left(\beta \sum_{m=1}^{n-i-j} b_{m+i+j} U(S_m)\right) \end{aligned}$$

$$\leqslant C \sum_{[(\mu(\beta)-\varepsilon)^{-1}\log n]\leqslant 2j\leqslant n-i} Z^{(0,n)}j^{1/2}\left[\left(E\exp\left(\beta\sum_{l=1}^{2j-1} b_{l+i}(U(S_l)-1)\right)I_{\{S_{2j}=0\}}\right)^{-1}\right.$$
$$\left.+\left(E\exp\left(\beta\sum_{l=1}^{2j-1} b_l(U(S_l)+1)\right)I_{\{S_{2j}=0\}}\right)^{-1}\right]$$

Thus, by Lemma 5.2 we can show that the l.h.s. of (5.2) is less than

$$C \sum_{[(\mu(\beta)-\varepsilon)^{-1}\log n]\leqslant 2j\leqslant n-i} Z^{(0,n)}j^{1/2}\left[\tilde{E}\left(E\exp\left(\beta\sum_{l=1}^{2j-1} b_l(U(S_l)-1)\right)I_{\{S_{2j}=0\}}\right)^{-1}\right.$$
$$\left.+\tilde{E}\left(E\exp\left(\beta\sum_{l=1}^{2j-1} b_l(U(S_l)+1)\right)I_{\{S_{2j}=0\}}\right)^{-1}\right]$$
$$\leqslant O(1)\, n^{-1-(\varepsilon/2)(\mu(\beta)-\varepsilon)^{-1}}$$

which implies the desired result. ▮

From Lemma 5.4 we can also see why $\mu(\beta)$ given by Lemma 5.2 is just the exponent related to the behavior of Δ_n which was described in Theorem 5.3.

To state the next two lemmas, let us first introduce some notations. For any given $\delta\in(0,\varepsilon/2)$ we set $u_n=n^\delta$ and $v_n=n^{1-\delta}$. Without loss of generality we may assume $u_n=[u_n]$ and $v_n=[v_n]$. Let

$$\xi(j)=\exp\left(\beta\sum_{i=1}^{ju_n} b_i U(S_i)\right) I_{\bigcap_{i=1}^{j}\{|S_{iu_n}|\leqslant(\mu(\beta)+\varepsilon)^{-1}\log n\}}$$

$$\eta_j(x_1,x_2)=E_{x_1} I_{\{\Delta_{u_n}\leqslant(\mu(\beta)-\varepsilon)^{-1}\log n\}}$$
$$\times\exp\left(\beta\sum_{i=1}^{u_n} b_{(j-1)u_n+i}U(S_i)\right) I_{\{S_{u_n}=x_2\}}$$

$$\zeta_j(x_1,x_2)=E_{x_1}\exp\left(\beta\sum_{i=1}^{u_n} b_{(j-1)u_n+i}U(S_i)\right) I_{\{S_{u_n}=x_2\}}$$

It is clear that $\eta_j\leqslant\zeta_j$. To complete the second part of the proof of Theorem 5.3 we should prove that the following holds with a positive probability:

$$\eta_j(x_1,x_2)\leqslant(1-n^{-1-\varepsilon})\,\zeta_j(x_1,x_2) \qquad (*)$$

for many x_1 and x_2. From this we can derive that the following holds with large probability:

$$\frac{\eta_j(x_1,x_2)}{\zeta_j(x_1,x_2)}\leqslant n^{-\varepsilon}, \qquad j=1,..., [n^{1-\delta}]$$

for many x_1 and x_2. This then implies Theorem 5.3. To prove $(*)$ we will give a decomposition of $\zeta_j(x_1, x_2) - \eta_j(x_1, x_2)$. For this purpose we set

$$I_1(n) = E_{x_1} \exp\left(\beta \sum_{i=1}^{[u_n/2]} b_{(j-1)u_n+i} U(S_i)\right) I_{\{S_{[u_n/2]}=0\}}$$

$$I_2(n) = E \exp\left(\beta \sum_{i=1}^{s_n} b_{(j-1)u_n+[u_n/2]+i} U(S_i)\right) I_{\{S_{s_n}=0\}}$$

$$I_3(n) = E \exp\left(\beta \sum_{i=1}^{u_n-[u_n/2]-s_n} b_{(j-1)u_n+[u_n/2]+s_n+i} U(S_i)\right) I_{\{S_{u_n-[u_n/2]-s_n}=x_2\}}$$

$$I_4(n) = E \exp\left(\beta \sum_{i=1}^{s_n} b_{(j-1)u_n+[u_n/2]+i} U(S_1)\right) I_{\{S_{s_n}=0\}}$$

$$I_n(n) = E_{x_1} \exp\left(\beta \sum_{i=1}^{[u_n/2]+s_n} b_{(j-1)u_n+i} U(S_i)\right) I_{\{S_{s_n+[u_n/2]}=0\}}$$

It is easy to see that $\zeta_j(x_1, x_2) - \eta_j(x_1, x_2) \geqslant I_1(n)\, I_3(n)\, I_4(n)$. We now give an estimate for $I_1(n)\, I_2(n)\, I_3(n)$ (see Lemma 5.5 below), which will lead to the proof of $(*)$.

Suppose $[u_n/2]$ is even. Otherwise, we may consider $[u_n/2]+1$ instead of $[u_n/2]$ below. Let $s_n = [(\mu(\beta)+\varepsilon)^{-1} \log n]$. Without loss of generality, we may assume s_n is even.

Lemma 5.5. For any given $\varepsilon \in (0, 1)$ and $K \geqslant 1$ there is a constant $C_3 \in (0, \infty)$ such that

$$\tilde{P}\left(\bigcup_{|x_1|, |x_2| \leqslant [(\mu(\beta)+\varepsilon)^{-1} \log n]} \{I_1(n)\, I_2(n)\, I_3(n) \leqslant \exp(-5\varepsilon^2 s_n)\, \zeta_j(x_1, x_2)\}\right)$$
$$\leqslant C_3 n^{-K}$$

Proof. Let

$$\tau = \sup\{i \leqslant [u_n/2]\colon S_i = 0\}$$

By a large-deviation result we can show that for any given $K \geqslant 1$ there is a constant $M > 0$ such that

$$\tilde{P}\left(E_{x_1} \exp\left(\beta \sum_{i=1}^{[u_n/2]+s_n} b_{(j-1)u_n+i} U(S_i)\right) I_{\{[u_n/2]-\tau \geqslant M s_n\}} \geqslant \frac{I_1(n)\, I_2(n)}{n}\right)$$
$$\leqslant O(1)\, n^{-K} \tag{5.3}$$

We remark that if $S_{s_n+[u_n/2]}=0$, then $|S_{[u_n/2]}|\leqslant s_n$. By Lemma 5.1 (replacing ε there by ε^2) and the Markov property we show that

$$
\begin{aligned}
&E_{x_1}\exp\left(\beta\sum_{i=1}^{[u_n/2]+s_n} b_{(j-1)u_n+i}U(S_i)\right) I_{\{S_{s_n+[u_n/2]}=0,\,[u_n/2]-\tau\leqslant Ms_n\}}\\
&\quad=\sum_{|x|\leqslant s_n} E_{x_1}\exp\left(\beta\sum_{i=1}^{[u_n/2]+s_n} b_{(j-1)u_n+i}U(S_i)\right)\\
&\qquad\times I_{\{S_{[u_n/2]}=x,\,S_{s_n+[u_n/2]}=0,\,[u_n/2]-\tau\leqslant Ms_n\}}\\
&\quad\leqslant\sum_{l_1=1}^{Ms_n}\sum_{|x|\leqslant s_n} E_{x_1}\exp\left(\beta\sum_{i=1}^{[u_n/2]-l_1} b_{(j-1)u_n+i}U(S_i)\right) I_{\{S_{[u_n/2]-l_1}=0\}}\\
&\qquad\times\sum_{l_2=1}^{s_n} E\exp\left(\beta\sum_{i=1}^{s_n-l_2} b_{(j-1)u_n+i+[u_n/2]}\,U(S_i)\right) I_{\{S_{s_n-l_2}=0\}}\\
&\qquad\times\exp\left(\left|\beta\sum_{i=[u_n/2]-l_1+1}^{[u_n/2]} b_{(j-1)u_n+i}\right|\right)\\
&\qquad\times\exp\left(\left|\beta\sum_{i=s_n-l_2+1}^{s_n} b_{(j-1)u_n+i+[u_n/2]}\right|\right)\\
&\quad\leqslant 8s_n\exp(2\varepsilon^2 s_n)\sum_{l_1=1}^{Ms_n} E_{x_1}\exp\left(\beta\sum_{i=1}^{[u_n/2]-l_1} b_{(j-1)u_n+i}U(S_i)\right)\\
&\qquad\times I_{\{S_{[u_n/2]-l_1}=0\}}\\
&\qquad\times\sum_{l_2=1}^{s_n} E\exp\left(\beta\sum_{i=1}^{s_n-l_2} b_{(j-1)u_n+i+[u_n/2(}\,U(S_i)\right) I_{\{S_{s_n-l_2}=0\}}\\
&\qquad\times E\exp\left(\beta\sum_{i=1}^{l_1} b_{(j-1)u_n+[u_n/2]-l_1+i}U(S_i)\right) I_{\{S_{l_1}=0\}}\\
&\qquad\times E\exp\left(\beta\sum_{i=1}^{l_2} b_{(j-1)u_n+s_n-l_2+i}U(S_i)\right) I_{\{S_{l_2}=0\}}\\
&\quad\leqslant 8s_n\cdot\exp(2\varepsilon^2 s_n)\,I_1(n)\,I_2(n)
\end{aligned}
$$

By (5.3) we know that

$$\tilde{P}(I_1(n)\,I_2(n)\leqslant(8s_n\exp(2\varepsilon^2 s_n))^{-1}(1-n^{-1})\,I_5(n))\leqslant O(1)\,n^{-K}$$

By a similar reasoning (using Lemma 5.1) we show that

$$\tilde{P}(I_5(n)\,I_3(n)\leqslant(8s_n\exp(2\varepsilon^2 s_n))^{-1}(1-n^{-1})\,\zeta(x_1,x_2))\leqslant O(1)\,n^{-K}$$

Thus we obtain

$$\tilde{P}(I_1(n)\, I_2(n)\, I_3(n) \leqslant (8s_n \exp(2\varepsilon^2 s_n))^{-2}(1-n^{-1})^2 \zeta(x_1, x_2))$$
$$\leqslant O(1)\, n^{-K}$$

Using this, we then complete the proof of Lemma 5.5. ▮

(∗) will be proven in the next lemma.

Lemma 5.6. For any given $\beta \neq 0$, $K \geqslant 1$, and $\varepsilon \in (0, 1 \wedge \mu(\beta))$ there are constants $C_4, C_5 \in (0, \infty)$ and $i_0 \in [-1, [\mu(\beta)/\varepsilon^2]+1]$ such that

$$\tilde{P}\left(\bigcap_{|x_1|,\, |x_2| \leqslant [(\mu(\beta)+\varepsilon)^{-1} \log n]} \left\{\frac{\eta_j(x_1, x_2)}{\zeta_j(x_1, x_2)} \leqslant 1 - C_4 \exp(-(\mu(\beta) - i_0 \varepsilon^2) s_n)\right\}\right)$$
$$\geqslant C_5(\exp(-(i_0+1)\,\varepsilon^2 s_n) - n^{-K})$$

Proof. By Lemma 5.2 we can show that for any given $\varepsilon \in (0, 1 \wedge \mu(\beta))$ there are a constant $C_6 \in (0, \infty)$ and $i_0 \in [-1, [\mu(\beta)/\varepsilon^2]+1]$ such that

$$\tilde{P}\Bigg(\exp(-(\mu(\beta) - (i_0-1)\,\varepsilon^2)\, s_n)$$
$$\leqslant \left(E \exp\left(\beta \sum_{i=1}^{s_n - 1} b_i(U(S_i)+1)\right) I_{\{S_{s_n}=0\}}\right)^{-1}$$
$$\leqslant \exp(-(\mu(\beta) - i_0\varepsilon^2)\, s^n)\Bigg)$$
$$\geqslant O(1) \exp(-(\mu(\beta)+\varepsilon^2)\, s_n) \cdot \exp((\mu(\beta) - i_0\varepsilon^2)\, s_n)$$
$$\geqslant C_6 \exp(-(i_0+1)\, \varepsilon^2 s_n)$$

By definition we know that

$$I_2^{-1}(n)\, I_4(n)$$
$$= \tfrac{1}{2}\left(E \exp\left(\beta \sum_{i=1}^{s_n} b_{(j-1)u_n + [u_n/2]+i}(U(S_i)+1)\right) I_{\{S_{s_n}=0\}}\right)^{-1}$$
$$\geqslant \tfrac{1}{2} e^{-2|\beta|}\left(E \exp\left(\beta \sum_{i=1}^{s_n - 1} b_{(j-1)u_n + [u_n/2]+i}(U(S_i)+1)\right) I_{\{S_{s_n}=0\}}\right)^{-1}$$

Hence,

$$\tilde{P}(\exp(-\mu(\beta) - (i_0-1)\,\varepsilon^2 s_n)) \leqslant I_2^{-1}(n)\, I_4(n)) \geqslant C_7 \exp(-(i_0+1)\,\varepsilon^2 s_n)$$

for some constant $C_7 \in (0, \infty)$. By the Markov property of $\{S_i\}$ we have

$$\zeta_j(x_1, x_2) - \eta_j(x_1, x_2)$$
$$\geqslant E_{x_1} I_{\{\Delta_{u_n} > s_n\}} \exp\left(\beta \sum_{i=1}^{u_n} b_{(j-1)u_n+i} U(S_i)\right) I_{\{S_{u_n} = x_2\}}$$
$$\geqslant I_1(n)\, I_2(n)\, I_3(n)(I_2^{-1}(n)\, I_4(n))$$

Thus, by Lemma 5.5 we have

$$\tilde{P}\left(\bigcap_{|x_1|,\, |x_2| \leqslant [(\mu(\beta)+\varepsilon)^{-1}\log n]} \left\{1 - \frac{\eta_j(x_1, x_2)}{\zeta_j(x_1, x_2)} \geqslant \exp[-\varepsilon^2 s_n - \mu(\beta) - (i_{0-1})_\varepsilon{}^2 s_n]\right\}\right)$$
$$\geqslant -C_6 n^{-K} + C_7 \exp[-(i_0+1)\, \varepsilon^2 s_n]$$

which proves Lemma 5.6. ∎

We are now in a position to complete the proof of Theorem 5.3.

Proof of Theorem 5.3. For any given $\varepsilon \in (0, 1 \wedge \mu(\beta))$, by Lemma 5.4 we know that

$$\tilde{E}P^{(0,n)}((\log n)^{-1} \Delta_n \geqslant (\mu(\beta) - \varepsilon)^{-1})$$
$$\leqslant \tilde{E}P^{(0,n)}\left(\bigcup_{i=1}^{n} \{S_i = 0, \sigma_i \geqslant (\mu(\beta) - \varepsilon)^{-1} \log n\}\right)$$
$$\leqslant \sum_{i=1}^{n} \tilde{E}P^{(0,n)}(S_i = 0, \sigma_i \geqslant (\mu(\beta) - \varepsilon)^{-1} \log n)$$
$$\leqslant O(1)\, n^{-(\varepsilon/2)(\mu(\beta)-\varepsilon)^{-1}}$$

which proves that

$$P^{(0,n)}((\log n)^{-1} \Delta_n \geqslant (\mu(\beta) - \varepsilon)^{-1}) \xrightarrow{\tilde{P}} 0, \qquad n \to \infty$$

To consider the other direction in the statement of the convergence in Theorem 5.3, we set

$$P_\varepsilon^{(0,n)}(A) = (E\xi(v_n))^{-1} \int_A \xi(v_n)\, dP, \qquad \forall A \in \mathscr{F}$$

and

$$\Delta_n(i) = \max_{(i-1)u_n < k < l \leqslant iu_n} \{l - k\colon S_l = S_k = 0,\ U(S_{l+1}) = \cdots = U(S_{k-1}) \neq 0\}$$

Then, by the Markov property of $\{S_i\}$ we can show that

$$
\begin{aligned}
&EI_{\{(\log n)^{-1}\Delta_n \leqslant (\mu(\beta)+\varepsilon)^{-1}\}}\,\xi(v_n)\\
&\quad\leqslant E\prod_{i=1}^{v_n} I_{\{\Delta_n(i)\leqslant(\mu(\beta)+\varepsilon)^{-1}\log n\}}\,\xi(v_n)\\
&\quad\leqslant \sum_{|x_1|,\dots,|x_{v_n}|\leqslant[(\mu(\beta)+\varepsilon)^{-1}\log n]} \eta_1(0,x_1)\,\eta_2(x_1,x_2)\cdots\eta_{v_n}(x_{v_n-1},x_{v_n})
\end{aligned}
\tag{5.4}
$$

Let

$$
A_j=\bigcap_{|x_1|,\,|x_2|\leqslant[(\mu(\beta)+\varepsilon)^{-1}\log n]}\left\{\frac{\eta_j(x_1,x_2)}{\zeta_j(x_1,x_2)}\leqslant 1-C_4e^{-(\mu(\beta)-i_0\varepsilon^2)s_n}\right\},\quad j=1,\dots,v_n
$$

where $i_0\in[-1,[\mu(\beta)/\varepsilon^2]+1]$ was given in Lemma 5.6. We can choose a large enough $K\geqslant 1$ such that

$$
n^{-K}\leqslant\tfrac{1}{4}e^{-(\mu(\beta)-i_0\varepsilon^2)s_n}
$$

Since $A_1,\dots,A_{v_n}$ are independent, by Lemma 5.6 we have

$$
\tilde{P}\left(\sum_{j=1}^{v_n} I_{A_j}<\tfrac{3}{8}C_5v_ne^{-(i_0+1)\varepsilon^2s_n}\right)\leqslant O(1)\,n^{-4}
$$

if $\delta\in(0,1)$ is small enough. If

$$
\sum_{j=1}^{v_n} I_{A_j}\geqslant\tfrac{3}{8}C_5v_ne^{-(i_0+1)\varepsilon^2s_n}
$$

has occurred, then there is a constant $C_8\in(0,\infty)$ such that r.h.s. of (5.4) is less than

$$
\begin{aligned}
&O(1)(1-C_4n^{-1+(\varepsilon+i_0\varepsilon^2)(\mu(\beta)+\varepsilon)^{-1}})\,n^{1-\delta-(i_0+1)\varepsilon^2(\mu(\beta)+\varepsilon)^{-1}}\\
&\qquad\times\sum_{|x_1|,\dots,|x_{v_n}|\leqslant[(\mu(\beta)+\varepsilon)^{-1}\log n]}\zeta_1(0,x_1)\,\zeta_2(x_1,x_2)\cdots\zeta_{v_n}(x_{v_n-1},x_{v_n})\\
&\quad\leqslant C_8n^{-4}E\xi(v_n)
\end{aligned}
$$

if $\delta\in(0,(\varepsilon-\varepsilon^2)/\{\mu(\beta)+\varepsilon\})$. From this we obtain that

$$
\tilde{P}(P_\varepsilon^{(0,n)}((\log n)^{-1}\Delta_n\leqslant(\mu(\beta)+\varepsilon)^{-1})>C_3n^{-2})\leqslant O(1)\,n^{-2}\tag{5.5}
$$

We now use (5.5) to prove

$$
P^{(0,n)}((\log n)^{-1}\Delta_n\leqslant(\mu(\beta)+\varepsilon)^{-1})\xrightarrow{\tilde{P}}0,\qquad n\to\infty\tag{5.6}
$$

In fact, it is clear that

$$|S_{iu_n}| \leqslant (\mu(\beta)+\varepsilon)^{-1} \log n, \qquad i=1,..., v_n$$

if $(\log n)^{-1} \Delta_n \leqslant (\mu(\beta)-\varepsilon)^{-1}$. Thus, l.h.s. of (5.6) is less than

$$\begin{aligned}&(EZ^{(0,n)})^{-1} E\xi(v_n) P_\varepsilon^{(0,n)}((\log n)^{-1} \Delta_n \leqslant (\mu(\beta)+\varepsilon)^{-1})\\ &\leqslant P_\varepsilon^{(0,n)}((\log n)^{-1} \Delta_n \leqslant (\mu(\beta)+\varepsilon)^{-1})\end{aligned}$$

From this and (5.5) we get indeed the desired result (5.6). This completes the proof of Theorem 5.3. ∎

Remark. The convergence stated in Theorem 5.3 is only proven to be true in probability. It would be very interesting to prove that such a convergence also holds almost surely. In fact, from the proof of Theorem 5.3 we see that

$$\lim_{n\to\infty} P^{(0,n)}((\log n)^{-1} \Delta_n \leqslant (\mu(\beta)+\varepsilon)^{-1})=0 \qquad \text{a.e.-}\tilde{P}$$

Unfortunately, we are presently unable to get an almost sure convergence in the other direction.

6. LOCALIZATION

In recent years there have been many studies on the localization of random walks and diffusion processes (see, e.g., refs. 2, 6, and 10 and references therein). In a similar spirit, in this section we shall investigate the localization of the random walk $\{S_n\}_{n\geqslant 0}$ under the probability measure $P^{(0,n)}$. Our main result in this section is as follows.

Theorem 6.1. (i) If $\mu(\beta)\in(0, 1/8]$, then the following holds for any given $\varepsilon\in(0, 1)$:

$$P^{(0,n)}\left((\log n)^{-1} \max_{1\leqslant i\leqslant n} |S_i| \in \left[\frac{1-\varepsilon}{(2\mu(\beta))^{1/2}}, \frac{1+\varepsilon}{(2\mu(\beta))^{1/2}}\right]\right) \xrightarrow{\tilde{P}} 1, \qquad n\to\infty$$

where $\mu(\beta)$ was defined in Lemma 5.2.

(ii) If $\mu(\beta)>1/8$, then the following holds for any given $\varepsilon\in(0, 1)$:

$$P^{(0,n)}\left((\log n)^{-1} \max_{1\leqslant i\leqslant n} |S_i| \in \left[\frac{1-\varepsilon}{1/4+2\mu(\beta)}, \frac{1+\varepsilon}{1/4+2\mu(\beta)}\right]\right) \xrightarrow{\tilde{P}} 1, \qquad n\to\infty$$

Let us first make a remark on the value of $\mu(\beta)$. By the Hölder inequality we can easily show that $\lim_{|\beta| \to \infty} \mu(\beta) = 0$ and $\lim_{|\beta| \to 0^+} \mu(\beta) = \infty$. Therefore, $\mu(\beta) \in (0, 1/8]$ if $|\beta|$ is large enough, and $\mu(\beta) > 1/8$ if $|\beta| > 0$ is sufficiently small. In other words, both cases in Theorem 6.1 can happen. The proof of Theorem 6.1 is also involved. The idea to prove Theorem 6.1 is basically similar to the one used in the proof of Theorem 5.3. So we will not give a detailed proof for Theorem 6.1. We will prove two lemmas.

Lemma 6.2. For $\beta \neq 0$ and any given $\varepsilon_1 \in (0, 1)$ we have

$$\lim_{n \to \infty} \tilde{E}P^{(0,n)}(\max_{1 \leqslant i \leqslant n} |S_i| \geqslant (\chi(\beta) + \varepsilon_1) \log n) = 0$$

where

$$\chi(\beta) = \begin{cases} (2\mu(\beta))^{-1/2} & \mu(\beta) \leqslant 1/8 \\ (1/4 + 2\mu(\beta))^{-1}, & \mu(\beta) > 1/8 \end{cases}$$

Remark. For the case of the ordinary random walk in $\mathbf{Z}^1$, it is well known that the quantity $n^{-1/2} \max_{1 \leqslant i \leqslant n} |S_i|$ is weakly convergent to a real random variable. This fact tells us that $\chi(\beta) = (2\mu(\beta))^{-1/2}$ for the case of the ordinary random walk in $\mathbf{Z}^1$. The random walk discussed in the present paper differs from an ordinary random walk by having an exponential term as a statistical weight, a fact which induces a different relation between $\chi(\beta)$ and $\mu(\beta)$, as given above.

Proof. For any given constant $C_0 \in (0, \infty)$ we set

$$g(x) = \inf_{y \geqslant 2x} \left\{ \frac{x^2}{2y} + C_0 y \right\} \tag{6.1}$$

It is easy to show that

$$g(x) = \begin{cases} (2C_0)^{-1/2} x, & C_0 \leqslant 1/8 \\ x(1/4 + 2C_0), & C_0 > 1/8 \end{cases}$$

For the simple random walk $\{S_i\}$ we have the following estimate [for sufficiently small $\varepsilon_2 \in (0, 1)$]:

$$P(\max_{1 \leqslant i \leqslant (j-1)\varepsilon_2 \log n} |S_i| \geqslant (\chi(\beta) + \varepsilon_1 - 2\varepsilon_2) \log n)$$

$$\leqslant O(1) \log n \exp\left(- \left(\frac{(\chi(\beta) + \varepsilon_1 - 2\varepsilon_2)^2}{2(j-1)\,\varepsilon_2} \log n \right) \right)$$

By Lemma 5.4 we know that for any given $\varepsilon_2 \in (0, 1)$ there is a constant $M \geqslant 1$ such that

$$\tilde{E}P^{(0,n)}(\Delta_n \geqslant (M\varepsilon_2 + \varepsilon_2) \log n) \leqslant O(1)\, n^{-2}$$

where Δ_n was defined at the beginning of Section 5. Thus, as in the proof of Lemma 5.4, by Lemma 5.2 and the Markov property of $\{S_i\}$ we can show that

$$\begin{aligned}
&\tilde{E}P^{(0,n)}\Big(\max_{1 \leqslant i \leqslant n} |S_i| \geqslant (\chi(\beta) + \varepsilon_1) \log n\Big) \\
&\leqslant O(1)\, n^{-2} + \sum_{i=1}^{n} \sum_{j=[2(\chi(\beta)+\varepsilon_1)/\varepsilon_2]}^{M+1} \tilde{E}P^{(0,n)}(S_i = 0;\ U(S_{u+i}) = 0 \\
&\qquad \text{for some } u \in [[(j-1)\varepsilon_2 \log n]],\ [(j+1)\, \varepsilon_2 \log n]]; \\
&\qquad \text{and } U(S_{v+i}) \neq 0 \\
&\qquad \text{for all } v \in [1, u-1], \\
&\qquad \max_{1 \leqslant u \leqslant [(j-1)\varepsilon_2 \log n]} |S_{u+i}| \geqslant (\chi(\beta) + \varepsilon_1 - 2\varepsilon_2) \log n) \\
&\leqslant O(1)\, n^{-2} + O(1)(M+1) n \log n \exp(-g(\chi(\beta) + \varepsilon_1 - 2\varepsilon_2) \log n)) \qquad (6.2)
\end{aligned}$$

where $g(x)$ was defined by (6.1) [replacing C_0 there by $\mu(\beta) - \varepsilon_2^2$]. It is easy to show that $g(\chi(\beta) + \varepsilon_1 - 2\varepsilon_2)$ is less than

$$\begin{cases} 1 - \left(\dfrac{\varepsilon_2^2}{2\mu(\beta)}\right)^{1/2} + (\varepsilon_1 - 2\varepsilon_2)(2(\mu(\beta) - \varepsilon_2^2))^{1/2}, & \mu(\beta) - \varepsilon_2^2 \leqslant 1/8 \\ 1 - \dfrac{2\varepsilon_2^2}{1/4 + 2\mu(\beta)} + \dfrac{\varepsilon_1 - 2\varepsilon_2}{1/4 + 2(\mu(\beta) - \varepsilon_2^2)}, & \mu(\beta) - \varepsilon_2^2 > 1/8 \end{cases}$$

Thus, if $\varepsilon_1 \in (2\varepsilon_2, 3\varepsilon_2)$ and $\varepsilon_2 \in (0, 1)$ is sufficiently small, we have that the l.h.s. of (6.2) is less than

$$O(1)\, n^{-2} + O(1)\, n \cdot n^{-1-\varepsilon_2/2} \leqslant O(1)\, n^{-\varepsilon_2/2}$$

which proves the desired result. ∎

To state the next lemma we introduce some notations. For any given $\delta \in (0, \varepsilon/2)$ we introduce $u_n = n^\delta$ and $v_n = n^{1-\delta}$ as in Section 5 and make the same assumptions on them as in that section. For any given $\varepsilon_3, \varepsilon_4 \in (0, 1)$ we set

$$q(j) = \frac{(\chi(\beta) - \varepsilon)^2}{2(j\varepsilon_3 + \varepsilon_3)} + (\mu(\beta) + \varepsilon_4)(j\varepsilon_3 + \varepsilon_3)$$

One can show that there is a $j_0 \in [[2(\chi(\beta)-\varepsilon)/\varepsilon_3], \infty)$ such that

$$q(j_0) = \inf_{j \geqslant [2(\chi(\beta)-\varepsilon)/\varepsilon_3]} q(j)$$

Let $t_n = [j_0 \varepsilon_3 \log n]$ and assume that t_n is even (otherwise, we may consider $t_n + 1$ rather than t_n below). Let

$$\eta_j'(x_1, x_2) = E_{x_1} I_{\{\max_{1 \leqslant i \leqslant u_n} |S_i| \leqslant (\chi(\beta)-\varepsilon)\log n\}} \times \exp\left(\beta \sum_{i=1}^{u_n} b_{(j-1)u_n+i} U(S_i)\right) I_{\{S_{u_n} = x_2\}}$$

The next lemma is an analog of Lemma 5.6.

Lemma 6.3. For any given $\beta \neq 0$ and suffuciently small $\varepsilon_5, \varepsilon_6 \in (0, 1)$ there are constants $C_1, C_2 \in (0, \infty)$ and $i_0 \in [-1, [\mu(\beta)/\varepsilon_5] + 1]$ such that

$$\tilde{P}\left(\bigcap_{|x_1|, |x_2| \leqslant (\chi(\beta)-\varepsilon)\log n} \left\{\frac{\eta_j'(x_1, x_2)}{\zeta_j(x_1, x_2)} \leqslant 1 - C_1 \exp(-\varepsilon_6 t_n - T_n)\right\}\right) \geqslant C_2 \exp(-i_0 + 1)\, \varepsilon_5 t_n)$$

where $\zeta_j(x_1, x_2)$ was defined in Section 5, and

$$T_n = \frac{(\chi(\beta)-\varepsilon)^2 (\log n)^2}{2t_n} + (\mu(\beta) + \varepsilon_4)\, t_n - (i_0 - 1)\, \varepsilon_5 t_n$$

Proof. As in the proof of Lemma 5.6, by Lemma 5.2 we can show that there are a constant $C_3 \in (0, \infty)$ and $i_0 \in [-1, [\mu(\beta)/\varepsilon_5] + 1]$ such that

$$\tilde{P}\left(E\left(\exp\left(\beta \sum_{i=1}^{t_n} b_i(U(S_i) + 1)\right) I_{\{S_{t_n} = 0\}}\right)^{-1} \in [\exp(-(\mu(\beta) - (i_0 - 1)\varepsilon_5)t_n), \exp(-(\mu(\beta) - i_0\varepsilon_5)t_n)]\right) \geqslant C_3 \exp(-(i_0 + 1)\, \varepsilon_5 t_n)$$

As for $I_2(n)$ and $I_3(n)$ given in Section 5, we introduce $I_2'(n)$ and $I_3'(n)$, replacing s_n there by t_n. Let

$$I_4'(n) = E \exp\left(\beta \sum_{i=1}^{t_n} b_{(j-1)u_n + [u_n/2] + i} U(S_i)\right) I_{\{S_{t_n} = 0\}} \times I_{\{\max_{1 \leqslant i \leqslant t_n} |S_i| > (\chi(\beta)-\varepsilon)\log n\}}$$

Then we show that

$$\zeta_j(x_1, x_2) - \eta'_j(x_1, x_2) \geqslant I_1(n)\, I'_2(n)\, I'_3(n)(I'^{-1}_2(n)\, I'_4(n))$$

As in the proof of Lemma 5.5, we obtain that for any given $\varepsilon_7 \in (0, 1)$ and $K \geqslant 1$ there is a constant $C_4 \in (0, \infty)$ such that

$$\tilde{P}\left(\bigcup_{|x_1|,\, |x_2| \leqslant (\chi(\beta) - \varepsilon) \log n} \{I_1(n)\, I'_2(n)\, I'_3(n) \leqslant \exp(-\varepsilon_7 t_n)\, \zeta_j(x_1, x_2)\}\right) \leqslant C_4 n^{-K}$$

We remark that

$$\begin{aligned} &P(\max_{1 \leqslant i \leqslant t_n} |S_i| > (\chi(\beta) - \varepsilon) \log n) \\ &\qquad \geqslant O(1) \exp\left(-\frac{(\chi(\beta) - \varepsilon)^2 (\log n)^2}{2t_n}\right) \end{aligned}$$

Then, as in the proof of Lemma 6.2, we show that

$$\tilde{P}(I'^{-1}_2(n)\, I'_4(n) \geqslant C_5 \exp(-T_n)) \geqslant C_6 \exp(-(i_0 + 1)\, \varepsilon_5 t_n)$$

for some constants C_5, $C_6 \in (0, \infty)$. Therefore, if $K \geqslant 1$ is large enough,

$$\begin{aligned} &\tilde{P}\left(\bigcap_{|x_1|,\, |x_2| \leqslant (\chi(\beta) - \varepsilon) \log n} \left\{1 - \frac{\eta'_j(x_1, x_2)}{\zeta_j(x_1, x_2)} \geqslant C_5 \exp(-\varepsilon_7 t_n - T_n)\right\}\right) \\ &\quad \geqslant \frac{1}{2} C_6 \exp(-(i_0 + 1)\, \varepsilon_5 t_n) \end{aligned}$$

which proves the desired result. ∎

We remark that there is a constant $\varepsilon_8 \in (0, 1)$, which is only related to ε, ε_3, and ε_4, such that

$$\frac{(\chi(\beta) - \varepsilon)^2 (\log n)^2}{2t_n} + (\mu(\beta) + \varepsilon_4)\, t_n \leqslant (1 - \varepsilon_8) \log n$$

Thus, as in the proof of Theorem 5.3, we can use Lemmas 6.2 and 6.3 to prove Theorem 6.1. We omit the details.

NOTE ADDED IN PROOF

It has been proven in ref. 13 that $\lim_{n \to \infty} p^{(0, n)} (S_k = x)$ exists with probability one for any fixed $k \geqslant 1$ and $x \in \mathbb{Z}^1$.

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